CONTRACTIONS AND FLIPS

FOR VARIETIES OF SMALL COMPLEXITY

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Introduction

Consider a projective, normal algebraic variety X over an algebraically closed field. In the study of morphisms $\varphi : X \to X'$ where X' is another projective, normal variety, a fundamental role is played by the "cone of effective one-cycles" $NE(X)$. Namely, the curves contracted by φ define a face F of $NE(X)$; moreover, φ can be recovered from F, provided that φ has connected fibers (then φ is the contraction of F). But it may happen that some faces of $NE(X)$ do not arise from morphisms; and the geometry of $NE(X)$ can be quite complicated, see e.g. [CKM] §4.

In the present paper, we prove that everything is fine for a class of varieties with group actions. More precisely, we consider a connected reductive group G acting on a projective, normal variety X. We assume that X is unirational, and that the complexity of the action is at most one, i.e. that a Borel subgroup of G has an orbit of codimension at most one in X. Then we prove that the convex cone $NE(X)$ is finitely generated, and that each of its faces can be contracted (1.3). Moreover, if X is $\mathbf{Q}$-factorial, then we can always flip bad contractions (1.4) and every sequence of directed flips is finite (2.5). It follows that for any closed subgroup H of G such that the complexity of G/H is at most one, there exists an equivariant completion X of G/H such that the opposite of the canonical divisor is ample (2.5). It is tempting to conjecture that the assumption on the complexity of G/H is not necessary.

Our results generalize work of the first author (see [B]) which concern spherical varieties, i.e. varieties of complexity zero. We also mention related work of L. Moser-Jauslin and T. Nakano on threefolds where the group SL(2) acts with a dense orbit (see [M] and [N]); these examples have complexity one.

Our proofs are based on two finiteness results. The first one asserts that the algebra of regular functions $\Gamma(X, \mathcal{O}_X)$ is finitely generated, whenever X is a normal, unirational G-variety of complexity at most one; see [K2]. For the second one, we consider a normal G-variety X of complexity at most one, and we prove that X has only finitely many equivariant completions $\overline{X}$, if we prescribe the valuations associated to all prime divisors in $\overline{X} \setminus X$; see 2.1-2.4.

Notation and terminology. We consider algebraic varieties and groups which are defined over a fixed algebraically closed field k. The field of rational functions on a variety X is denoted by $k(X)$. We denote by G a connected reductive group; we choose a Borel subgroup B of G, and a maximal torus T of G. A *G-variety* X is a variety endowed with an action of G; then the *complexity* of X is the minimal codimension of a B-orbit in X; see [V]. The complexity of X is equal to the transcendence degree of $k(X)^B$ over k, where $k(X)^B$ denotes the subfield of B-invariants in $k(X)$.

Consider two varieties X and S, and a proper morphism $f : X \to S$. For any line bundle $\mathcal{L}$ over X, and for any (reduced and irreducible) complete curve C in X, we denote by $(\mathcal{L} \cdot C)$ the degree of the restriction of $\mathcal{L}$ to C. Denote by $Z_1(X/S)$ the free abelian group generated by all closed curves C in X such that $f(C)$ is a point; denote by $Pic(X/S)$ the quotient of $Pic(X)$ by $f^*Pic(S)$. Then the assignement $(\mathcal{L}, C) \to (\mathcal{L} \cdot C)$ defines a bilinear form

$$Pic(X/S) \times Z_1(X/S) \to \mathbf{Z} .$$

Dividing by the kernels and tensoring by $\mathbf{Q}$, we obtain a non-degenerate pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbf{Q}$$

where $N^1(X/S)$ (resp. $N_1(X/S)$) is the space of relative line bundles (resp. one-cycles), with rational coefficients, modulo numerical equivalence. We denote by $NE(X/S)$ the convex cone of $N_1(X/S)$ which is generated by the classes of closed curves C in X, such that $f(C)$ is a point.

Let $\mathcal{L}$ be a line bundle over X. Then $\mathcal{L}$ is called *f-nef* if $(\mathcal{L} \cdot C) \geq 0$ for any curve C in X such that $f(C)$ is a point. Equivalently, the linear form on $N_1(X/S)$ defined by $\mathcal{L}$ is non-negative on $NE(X/S)$. On the other hand, $\mathcal{L}$ is called *f-semi-ample* if there exists an integer $n > 0$ such that the natural homomorphism $f^*f_*(\mathcal{L}^{\otimes n}) \to \mathcal{L}^{\otimes n}$ is surjective. Observe that any f-semi-ample line bundle is f-nef. The converse is not true in general, but it holds whenever X is unirational and has complexity at most one; see 1.2.

1. Existence of contractions and of flips

1.1. For later purpose, we need the following characterization of semi-ample divisors among nef divisors, which may be of independant interest.

Proposition. *Consider a projective morphism $f : X \to S$ between normal varieties, and a f-nef line bundle $\mathcal{L}$ over X. Then the following conditions are equivalent:*
(i) $\mathcal{L}$ is f-semi-ample.
(ii) For any f-ample line bundle $\mathcal{M}$ over X, the sheaf of algebras

$$A(\mathcal{L}, \mathcal{M}) := \bigoplus_{l,m \geq 0} f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m})$$

is finitely generated over $\mathcal{O}_S$.
(iii) There exists a f-ample line bundle $\mathcal{M}$ over X, such that $A(\mathcal{L}, \mathcal{M})$ is finitely generated over $\mathcal{O}_S$.

Proof. $(i) \Rightarrow (ii)$ Denote by $\check{\mathcal{L}}$ (resp. $\check{\mathcal{M}}$) the total space of the dual bundle of $\mathcal{L}$ (resp. $\mathcal{M}$). Consider the vector bundle $\check{\mathcal{L}} \oplus \check{\mathcal{M}}$ over X, and the associated projective bundle $\pi : \mathbf{P} \to X$. Set $g = f \circ \pi$. We have the tautological line bundle $\mathcal{O}_{\mathbf{P}}(1)$ over $\mathbf{P}$, such that $\pi_* \mathcal{O}_{\mathbf{P}}(1) = \mathcal{L} \oplus \mathcal{M}$. So for any integer $n \geq 0$, we have:

$$g_* \mathcal{O}_{\mathbf{P}}(n) = \bigoplus_{0 \leq l \leq n} f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes (n-l)})$$

and therefore:

$$A(\mathcal{L}, \mathcal{M}) = \bigoplus_{n=0}^{\infty} g_* \mathcal{O}_{\mathbf{P}}(n) \ .$$

So for $A(\mathcal{L}, \mathcal{M})$ to be finitely generated over $\mathcal{O}_S$, it is enough to show that the line bundle $\mathcal{O}_{\mathbf{P}}(1)$ is g-semi-ample. But this follows from the f-semi-ampleness of $\mathcal{L}$, and the f-ampleness of $\mathcal{M}$.
$(iii) \Rightarrow (i)$ We may assume that S is affine; then we have to show that $\mathcal{L}$ is semi-ample. Choose a point $x \in X$. The $\mathbf{N}^2$-graded algebra

$$\bigoplus_{l,m \geq 0} \Gamma(\{x\}, \mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m})$$

can be identified with the polynomial algebra $k[u, v]$ where the degree of u (resp. v) is (1,0) (resp. (0,1)). The evaluation at x defines a morphism of $\mathbf{N}^2$-graded algebras

$$e_x : A(\mathcal{L}, \mathcal{M}) \to k[u, v] \ .$$

Because the algebra $\mathcal{A}(\mathcal{L}, \mathcal{M})$ is finitely generated, the set of all degrees occuring in $e_x(A(\mathcal{L}, \mathcal{M}))$ is a finitely generated semigroup. Choose non-zero generators $(l_1, m_1), \ldots, (l_t, m_t)$ of this semigroup with $l_i m_{i+1} - l_{i+1} m_i > 0$ for $1 \leq i \leq t-1$. If $m_1 \neq 0$ then $e_x(A(\mathcal{L}, \mathcal{M}))_{l,m} = 0$ for any (l, m) such that $l m_1 - l_1 m > 0$. Choose such a couple (l, m) with $m > 0$. Then the line bundle $\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m}$ is ample (because $\mathcal{L}$ is nef and $\mathcal{M}$ is ample), but all sections of all powers of this line bundle vanish at x, a contradiction. So $m_1 = 0$, and $\mathcal{L}^{\otimes l_1}$ has global sections which do not vanish at x.

1.2. Theorem. *Let $f : X \to S$ be a proper G-morphism between normal G-varieties. Assume that X is unirational and of complexity at most one. Then every f-nef line bundle over X is f-semi-ample.*

Proof. By standard reductions based on [S] Theorem 4.9, we may assume that the morphism f is projective. Let $\mathcal{L}$ be a f-nef line bundle over X. By replacing $\mathcal{L}$ with some positive power, we may assume that $\mathcal{L}$ is G-linearized. Choose a G-linearized, f-ample line bundle $\mathcal{M}$ over X.

By [K1] §2, we can cover S by translates of B-stable affine open subsets. Choose such a subset S_0. We have to show that the algebra

$$\bigoplus_{l,m\geq 0} \Gamma(S_0, f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m}))$$

is finitely generated. For this, we may assume that $S = G \cdot S_0$. Then $D := S \setminus S_0$ is a Cartier divisor of S; see [K3] Lemma 2.2. There exists a positive integer N such that the line bundle $\mathcal{O}_S(ND)$ is G-linearized. Set $\mathcal{N} := f^*\mathcal{O}_S(ND)$. Then the group $\hat{G} := G \times (\mathbf{G}_m)^3$ acts on the variety

$$\hat{X} := \mathrm{Spec}_{\mathcal{O}_X} \bigoplus_{l,m,n\geq 0} \mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m} \otimes \mathcal{N}^{\otimes n} .$$

Moreover, $\hat{X}$ is a normal, unirational $\hat{G}$-variety of complexity at most one. By [K2], the algebra $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ is finitely generated. Therefore, the algebra

$$\bigoplus_{l,m,n\geq 0} \Gamma(S, f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m}) \otimes \mathcal{O}_S(nND))$$

is, too. So the same holds for the algebra

$$\bigoplus_{l,m\geq 0} \Gamma(S_0, f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m})) = \bigcup_{n\geq 0} \bigoplus_{l,m\geq 0} \Gamma(S, f_*(\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m}) \otimes \mathcal{O}_S(nND)) .$$

We conclude by 1.1.

1.3. Theorem. *Let $f : X \to S$ be a projective G-morphism between normal G-varieties. Assume that X is unirational and of complexity at most one.*
(i) The cone $NE(X/S)$ is polyhedral, and each of its extremal rays is generated by the class of a B-stable, rational curve.
(ii) For any face F of $NE(X/S)$, there exists a unique normal G-variety X_F, projective over S, and a unique G-morphism $cont_F : X \to X_F$ with connected fibers, such that $F = NE(X/X_F)$. Moreover, F generates the kernel of $(cont_F)_* : N_1(X/S) \to N_1(X_F/S)$; and the space $N^1(X_F/S)$ is identified with the orthogonal of F in $N^1(X/S)$.
(iii) If $\varphi : X \to X'$ is any morphism to a projective variety over S, such that F is contained in $NE(X/X')$, then φ factorizes through $cont_F$.

Proof. (i) It follows from [M] Lemma 6.1 that any effective cycle of X which is contracted by f, is rationally equivalent to a B-stable effective cycle which is contracted by f. Therefore, it is enough to show that the B-stable irreducible curves of X which are contracted by f are rational, and that their images in $NE(X/S)$ generate only finitely many half-lines. Let C be such a curve. If B acts non-trivially on C, then C is obviously rational. Moreover, C^T consists in exactly

2 points, and the image of the half-line $\mathbf{Q}^+C$ in $NE(X/S)$ only depends on the connected components of X^T which meet C (see [B] 1.6). On the other hand, if B acts trivially on C, then there exists a unique parabolic subgroup P containing B which is opposite to the isotropy subgroups of all points in a non-empty open subset of C. By [K1] 1.2, we can choose a P-stable open affine subset X_0 of X meeting C, such that the quotient $\pi : X_0 \to X_0/P^u$ exists. Therefore, the restriction of π to $C \cap X_0$ is injective. We set: $L := P/P^u$ and $\Sigma := X_0/P^u$. Observe that Σ is an affine, unirational L-variety of complexity at most one; hence its (Mumford) quotient Σ/L is a point or a rational, irreducible curve. But $\pi(C \cup X_0)$ is a curve in Σ^L; moreover, the composition $\Sigma^L \to \Sigma \to \Sigma/L$ is injective. Therefore, the composition $\pi(C \cap X_0) \to \Sigma/L$ is bijective. It follows that $\pi(C \cap X_0)$ is rational, and that C is rational, too.

(ii) and (iii) are formal consequences of 1.2 (see [KMM] 3.2.5, [B] 3.1).

1.4. Let X be a $\mathbf{Q}$-factorial, unirational G-variety of complexity at most one. Let $f : X \to S$ be a projective G-morphism; let R be an extremal ray R of $NE(X/S)$ By 1.3, the contraction of R exists; denote it by $\varphi : X \to X'$. We assume that φ is an isomorphism in codimension at most one.

Proposition. *Under the assumptions above, there exists a unique $\mathbf{Q}$-factorial G-variety X^+, projective over S, and a unique G-morphism $\varphi^+ : X^+ \to X'$ such that:*
(i) φ^+ is the contraction of an extremal ray R^+ of $NE(X^+/S)$.
(ii) φ^+ is an isomorphism in codimension at most one.
(iii) If the spaces $N^1(X/S)$ and $N^1(X^+/S)$ are identified via $\varphi^+ \circ \varphi^{-1}$, then the half-lines R and R^+ are opposite in $N_1(X/S) = \mathrm{Hom}(N^1(X/S), \mathbf{Q})$.
We call $\varphi^+ : X \to X^+$ the *flip* of φ.

Proof. By [KMM] Proposition 5.1.11, the statement is a consequence of the following assertion, whose proof (analogous to 1.2) is left to the reader: For any line bundle $\mathcal{L}$ on X, the sheaf of algebras $\oplus_{n=0}^{\infty} \varphi_*(\mathcal{L}^{\otimes n})$ is finitely generated over S.

2. Termination of flips

2.1. Let X be a homogeneous G-variety. Denote by $\mathcal{V}$ the set of all G-invariant k-valuations of the field $k(X)$ with values in $\mathbf{Q}$. For any equivariant normal embedding $\overline{X}$ of X, denote by $\mathcal{D}(\overline{X})$ the set of all G-stable prime divisors in $\overline{X}$. We identify a prime divisor $D \subset \overline{X}$ and the associated (normalized) valuation v_D of $k(\overline{X}) = k(X)$, so $\mathcal{D}(\overline{X})$ is a finite subset of $\mathcal{V}$.

Theorem. *Let X be a homogeneous G-variety of complexity at most one. Let $\mathcal{D}$ be a finite subset of $\mathcal{V}$. Then there exist only finitely many complete normal embeddings $\overline{X}$ with $\mathcal{D}(\overline{X}) = \mathcal{D}$.*

2.2. Before we enter the proof we need some preparation. Denote by $\mathcal{F}$ the set of all B-stable prime divisors in X. For any G-stable subvariety Y in $\overline{X}$ define

$$\mathcal{V}_Y(\overline{X}) := \{D \in \mathcal{D}(\overline{X}) \mid Y \subset D\};$$

$$\mathcal{F}_Y(\overline{X}) := \{D \in \mathcal{F} \mid Y \subset \overline{D}\};$$

$$\mathbf{F}_Y(\overline{X}) := (\mathcal{V}_Y(\overline{X}), \mathcal{F}_Y(\overline{X})).$$

So the pair $\mathbf{F}_Y(\overline{X})$ describes the set of B-stable divisors of $\overline{X}$ which contain Y. We recall that the embedding $\overline{X}$ is uniquely determined by

$$\mathbf{F}(\overline{X}) := \{\mathbf{F}_Y(\overline{X}) \mid Y \subset \overline{X} \text{ closed orbit}\}$$

(see [K1] 3.8). This immediately implies Theorem 2.1 when $c(X) = 0$, because $\mathcal{F}$ is finite in this case. Therefore, we assume from now on that $c(X) = 1$, i.e. that the transcendence degree of $k(X)^B$ over k is one.

Let C be the smooth projective curve with $k(C) = k(X)^B$. The points of C can be identified with the equivalence classes of non-trivial valuations of $k(X)^B$. Let v_o be the trivial valuation. Then we can break up $\mathcal{V}$ and $\mathcal{F}$ into pieces, as follows. For any $c \in C \cup \{o\}$, we set (with $0v_c := v_o$):

$$\mathcal{V}_c := \{v \in \mathcal{V} \mid v|_{k(C)} \in \mathbf{Q}^{\geq 0} v_c\};$$

$$\mathcal{F}_c := \{D \in \mathcal{F} \mid v_D|_{k(C)} \in \mathbf{Q}^{>0} v_c\}.$$

Observe that $\mathcal{V}_c \cap \mathcal{V}_d = \mathcal{V}_o$ for any distinct c, d in $C \cup \{0\}$. Let $\mathcal{O}_c$ be the valuation ring of v_c in $k(C)$. Consider the $\mathbf{Q}$-vector space

$$\mathcal{Q}_c := \mathrm{Hom}(k(X)^{(B)}/\mathcal{O}_c^\times, \mathbf{Q}).$$

Then $\mathcal{Q}_c$ is finite-dimensional (see [K1] §5). Moreover, $\mathcal{Q}_o$ is a hyperplane in $\mathcal{Q}_c$ for $c \neq 0$. Restriction to $k(X)^{(B)}$ defines maps

$$\mathcal{V}_c \to \mathcal{Q}_c; \ \rho : \mathcal{F}_c \to \mathcal{Q}_c.$$

The first one is injective ([K1] 3.6) and we will identify $\mathcal{V}_c$ and its image in $\mathcal{Q}_c$.

Lemma. *Let $c \in C \cup \{o\}$.*
a) The set $\mathcal{V}_c$ is a finitely generated convex cone.
b) If $c \neq o$ then $\mathcal{V}_o$ is a 1-codimensional face of $\mathcal{V}_c$.
c) The set $\mathcal{F}_c$ is finite.
d) There is a non-empty open subset C^0 of C such that $\mathcal{F}_d$ consists in exactly one divisor D_d whenever $d \in C^0$.
e) There exists a non-empty open subset C^1 of C^0 such that $\mathcal{V}_d$ is contained in the convex cone generated by $\rho(D_d)$ and $\mathcal{V}_o$ whenever $d \in C^1$.

Proof. For a) and b) see [K1] 6.5. We may choose a non-empty, B-stable open subset X_0 of X, such that the orbit space X_0/B exists, with quotient map π. Moreover, we may identify X_0/B with an open subset C^0 of C. If $D \in \mathcal{F}_c$ meets X_0 then D is the closure of $\pi^{-1}(c)$; denote it by D_c. Otherwise, D is one of the finitely many components of $X \setminus X_0$. This implies c) and d).

To prove e), we construct a certain embedding of X. Because X is homogeneous, C is unirational. By Lüroth's theorem, there exists $t \in k(C)$ such that $k(C) = k(t)$. The choice of t identifies C with $\mathbf{P}^1$. Denote by D_0 the divisor on X

$$(t)_\infty + \sum_{D \in \mathcal{F}_o} D$$

Set $\mathcal{L} := \mathcal{O}_X(D_0)$, and denote by σ_0 the canonical section of $\mathcal{L}$. Then $\sigma_1 := t\sigma_0$ is a section as well. By replacing G with a finite cover we may assume that $\mathcal{L}$ is G-linearized. Let M be the G-submodule of $\Gamma(X, \mathcal{L})$ generated by σ_0 and σ_1. Let $\overline{X}$ be an equivariant normal, complete embedding such that $\mathcal{L}$ extends to $\overline{X}$ and that the linear system M has no base point in $\overline{X}$.

Set $\overline{X}_0 := \{x \in \overline{X} \mid \sigma_0(x) \neq 0\}$. Then $t = \sigma_1/\sigma_0$ defines a B-invariant morphism $\tau : \overline{X}_0 \to \mathbf{A}^1 \subset \mathbf{P}^1 = C$. The generic fiber of τ is connected because $k(t) = k(X)^B$ is algebraically closed in $k(X)$. Now let C^1 be the set of all $c \in C^0 \cap \mathbf{A}^1$ such that $\tau^{-1}(c)$ is non-empty and irreducible, and meets X.

We check that the lemma holds for C^1. Let $c \in C^1$. Then $\overline{\tau^{-1}(c)}$ is an irreducible divisor, stable by B but not by G. Hence $\overline{\tau^{-1}(c)}$ is equal to D_c. Now choose $v \in \mathcal{V}_c$. Then $c \in \mathbf{A}^1$ means $v(t) \geq 0$ and this implies $v(M/\sigma_0) \geq 0$ by [K1] 3.3. Let Z be the center of v in $\overline{X}$. Because M is base point free, σ_0 cannot vanish on Z, i.e. Z meets $\overline{X}_0$. Moreover, $v \in \mathcal{V}_c$ implies $\tau(Z \cap \overline{X}_0) = \{c\}$. Therefore, D_c is the only B-stable prime divisor which contains Z and which is not mapped dominantly to C by τ. Hence we get $\mathcal{V}_Z(\overline{X}) \subset \mathcal{V}_o$ and $\mathcal{F}_Z(\overline{X}) = \{D_c\}$ because, by definition of D_0, no $D \in \mathcal{F}_o$ meets $\overline{X}_0$.

Assume that v is not in the convex cone generated by $\rho(D_c)$ and $\mathcal{V}_o$. Then there exists $f \in k(X)^{(B)}$ such that $v(f) < 0$ but $v_D(f) \geq 0$ for any B-stable prime divisor D which contains Z. But this contradicts the fact that Z is the center of v.

2.3. *Proof of Theorem 2.1:* Define a map $\zeta : \mathcal{V} \to C \cup \{o\}$ by $\zeta(\mathcal{V}_o) = \{o\}$ and $\zeta(\mathcal{V}_c \setminus \mathcal{V}_o) = \{c\}$. Choose $C^1 \subset C$ as in Lemma 2.2 and set

$$C^2 := C^1 \setminus \zeta(\mathcal{D}),\ S := C \setminus (C^2 \cup \{o\}) \text{ and } \mathcal{F}' := \bigcup_{c \in S} \mathcal{F}_c\ .$$

Observe that $\mathcal{F}'$ is finite. We consider sets of couples (V, F) such that $V \subset \mathcal{V}$ and $F \subset \mathcal{F}$. We call such a set *$\mathcal{D}$-admissible* if it is the union of sets which appear in the following list:

A) $\{(V, F)\}$ for some $V \subset \mathcal{D}$ and $\mathcal{F} \setminus \mathcal{F}' \subset F \subset \mathcal{F}$.

B) $\{(V,F)\}$ for some $V\subset\mathcal{D}$ and $F\subset\mathcal{F}'$.

C) $\{(V,F'\cup\{D_c\})\mid c\in C^2\}$ for some $V\subset\mathcal{D}\cap\mathcal{V}_o$ and $F'\subset\mathcal{F}_o$.

The admissible sets of types A and B consist of a single element, while those of type C are infinite. Observe that there are only finitely many $\mathcal{D}$-admissible subsets for prescribed $\mathcal{D}$, due to the fact that $\mathcal{D}$, $\mathcal{F}'$ and $\mathcal{F}_o$ are finite. Now Theorem 2.1 results from the following

Lemma. *Let $X\subset\overline{X}$ be a complete normal embedding with $\mathcal{D}(\overline{X})=\mathcal{D}$. Then the set $\mathbf{F}(\overline{X})$ is $\mathcal{D}$-admissible.*

Proof of the lemma. Let Y be a closed G-orbit in $\overline{X}$. Let P be the parabolic subgroup of G containing B which is opposite to some isotropy subgroup of G in Y. By [K1] 1.2, there exists a P-stable open affine subset $\overline{X}_0$ of $\overline{X}$ meeting Y, such that the quotient $\pi:\overline{X}_0\to\overline{X}_0/P^u$ exists. It follows that $\pi(\overline{X}_0\cap Y)$ is a point, which we denote by y. Moreover, y is a fixed point of P in $\overline{X}_0/P^u:=\Sigma$.

The equality $k(\Sigma)^B=k(X)^B=k(C)$ induces a B-invariant rational map $f:\Sigma-\to C$. Denote by Σ' the normalization of the closure of the graph of f. Then we have a morphism $f':\Sigma'\to C$ and a proper, birational morphism $p:\Sigma'\to\Sigma$ such that $f'=f\circ p$.

If f is not defined at y then f' maps $p^{-1}(y)$ onto C. For any $c\in C$ choose a component Σ_c of $p(f'^{-1}(c))$ containing y. Then $\overline{X}_c:=\overline{\pi^{-1}(\Sigma_c)}$ is a B-stable divisor of $\overline{X}$ containing Y. It induces on $k(C)$ a valuation which is equivalent to v_c. If $\overline{X}_c$ is G-stable, then $c\in\zeta(\mathcal{D})$. Hence $c\in C^2$ implies $\overline{X}_c\in\mathcal{F}_c=\{D_c\}$, i.e. $\mathbf{F}_Y(\overline{X})$ is of type A.

If f is defined at y, then we set $c:=f(y)$. Let D be a B-stable prime divisor in Σ containing y. Then either $f(D)$ is dense in C, or $f(D)=\{c\}$. This implies $\mathcal{F}_Y(\overline{X})\subset\mathcal{F}_o\cup\mathcal{F}_c$. If moreover $c\notin C^2$ then the set $\mathbf{F}_Y(\overline{X})$ is of type B, and we are done.

Assume from now on that $c\in C^2$. Then $c\notin\zeta(\mathcal{D})$ implies that no component of $\overline{(f\circ\pi)^{-1}(c)}$ is G-stable. Because $\mathcal{F}_c=\{D_c\}$, we have $\mathcal{V}_Y(\overline{X})\subset\mathcal{D}\cap\mathcal{V}_o$ and $\mathcal{F}_Y(\overline{X})=F\cup\{D_c\}$ for some $F\subset\mathcal{F}_o$. Therefore, $\mathbf{F}_Y(\overline{X})$ is an element of a $\mathcal{D}$-admissible set of type C. We have to prove that all other elements of this set are in $\mathbf{F}(\overline{X})$.

First we claim that f is actually P-invariant. Namely, let $d\neq c$ be in the image of f. Then $D:=f^{-1}(d)$ is a B-stable divisor with $y\notin D$. Because P/B is complete, PD is closed in Σ. Moreover, $y\notin PD$. For dimension reasons, this implies $PD=D$ and the claim.

Let $\mathcal{C}\subset\mathcal{Q}_c$ be the convex cone spanned by $\mathcal{V}_Y(\overline{X})$ and $\rho(\mathcal{F}_Y(\overline{X}))$. Set $\mathcal{C}_o:=\mathcal{C}\cap\mathcal{Q}_o$. Then $\mathcal{C}$ is generated by $\rho(D_c)$ and $\mathcal{C}_o$. Choose a valuation $v\in\mathcal{V}$ with center Y. We can write $v=a\rho(D_c)+v_o$ with $a\in\mathbf{Q}^{>0}$ and $v_o\in\mathcal{C}_o$. Then Lemma 2.2 e) implies that $v_o\in\mathcal{V}_o$. Let $Z\subset\overline{X}$ be the center of v_o. Then $v_o\in\mathcal{C}$ implies $v_o(\mathcal{O}_{\overline{X},Y})\geq 0$ and therefore $Y\subset Z$ by [K1] 3.7.

Set $Q:=\pi(Z\cap\overline{X}_0)$, and $W:=Q\cap f^{-1}(c)$; then $y\in W$. We claim that $W=\{y\}$. Otherwise, there exists $h\in k[\Sigma]^{(B)}$ with $h(y)\geq 0$ and $h|_W\neq 0$ ([K1] 2.2 applied to the action of P/P_u on Σ). But this implies $v_o(h)>0$ which is absurd.

Because $v_o \in \mathcal{V}_o$, the restriction $f|_Q$ is dominant. By the claim, $f|_Q$ is quasifinite. Therefore $Q \subset \Sigma^L$. So we have $k(Z)^{(B)} = k(Q) = k(Z)^B$, which implies that all G-orbits in Z are closed ([K1] 8.5). Moreover, we have $\mathcal{V}_Z(\overline{X}) = \mathcal{V}_Y(\overline{X})$ and $\mathcal{F}_Z(\overline{X}) = F = \mathcal{F}_Y(\overline{X}) \setminus \{D_c\}$.

The restriction of $f \circ \pi$ to Y_0 induces a rational G-invariant map $Z - \to C$ which is regular on the normalization $\tilde{Z}$ of Z (observe that all G-orbits in Z are closed of codimension one). Because $\overline{X}$ is complete, the induced map $\tilde{Z} \to C$ is surjective and its fibres are exactly the G-orbits. For $d \in C^2$ let Y_d be the image in Z of the orbit over d. Now the discussion above with Y replaced by Y_d shows that $\mathcal{F}_{Y_d}(\overline{X})$ is an element of a set of type C, and hence $\mathcal{F}_{Y_d}(\overline{X}) = (V, F \cup \{D_d\}$ with $V = \mathcal{V}_Z(\overline{X})$ and $F = \mathcal{F}_Z(\overline{X})$ independant of d. This ends the proof of Lemma 2.3.

2.4. There is a generalization to the case where X is any normal G-variety of complexity one. An *equivariant model* of X is a normal G-variety $\overline{X}$ together with a birational equivariant map $X - \to \overline{X}$.

Theorem. *Let X be a normal G-variety of complexity at most one. Let $\mathcal{D}$ be a subset of $\mathcal{V}$. Then there exist only finitely many complete normal equivariant models $\overline{X}$ of X with $\mathcal{D}(\overline{X}) = \mathcal{D}$.*

Proof. We may assume that X does not contain a dense G-orbit. We will only sketch the proof because it goes along the same lines of that of Theorem 2.1 with the roles of $\mathcal{V}$ and $\mathcal{F}$ being exchanged. Here $\mathcal{F}$ is the set of B-stable prime divisors of X which are not G-stable. So $\mathcal{F}$ depends only on the birational class of X. Then the definitions of $\mathcal{V}_Y(\overline{X})$, $\mathcal{D}_Y(\overline{X})$, $\mathbf{F}_Y(\overline{X})$ go through, and $\overline{X}$ is uniquely determined by the collection of all $\mathbf{F}_Y(\overline{X})$.

By assumption we have $k(X)^G \neq k$. It follows that $k(X)^G = k(X)^B = k(C)$ where C is a uniquely defined smooth, projective curve. The definitions of $\mathcal{V}_c$, $\mathcal{F}_c$ and $\mathcal{Q}_c$ for $c \in C \cup \{o\}$ are the same as in the homogeneous case, and parts a) and b) of Lemma 2.2 hold verbatim.

By making X smaller, we may assume that the rational map $f : X - \to C$ is regular, and that the fibers of f are G-orbits. Set $C^0 := f(X)$. Then for every $c \in C^0$ the fiber $f^{-1}(c)$ is a prime divisor which induces a normalized valuation $v_c \in \mathcal{V}_c$. This also shows that $\mathcal{F}_c$ is empty unless $c = o$ in which case it is finite. Now Lemma 2.2 e) has the following analogue with a similar proof.

Lemma. *There is a non-empty open subset $C^1 \subset C^0$ such that $\mathcal{V}_c$ is the convex cone spanned by $\mathcal{V}_o$ and v_c for every $c \in C^1$.*

Now let $\overline{X}$ be any complete equivariant model of X with $\mathcal{D}(\overline{X}) = \mathcal{D}$. Then the rational map $\overline{f} : \overline{X} - \to C$ is defined on a G-stable open subset which contains X. Therefore, the sets $\mathcal{D}$ and $\{v_c \mid c \in C^0\}$ coincide up to a finite set. For $c \in C \cup \{0\}$ let $\mathcal{D}_c := \mathcal{D} \cap \zeta^{-1}(c)$. This set is finite. We define

$$C^2 := \{c \in C^1 \mid \mathcal{D}_c = \{v_c\}\},\ S := C \setminus (C^2 \cup \{o\}),\ \mathcal{D}' := \bigcup_{c \in S} \mathcal{D}_c.$$

Observe that $\mathcal{D}'$ is finite. We define a $\mathcal{D}$-admissible set as a set of pairs (V, F) which is the union of sets appearing in the following list:

A) $\{(V, F)\}$ for some $\mathcal{D} \setminus \mathcal{D}' \subset V \subset \mathcal{D}$ and $F \subset \mathcal{F}$.

B) $\{(V, F)\}$ for some $V \subset \mathcal{D}'$ and $F \subset \mathcal{F}$.

C) $\{(V \cup \{v_c\}, F) \mid c \in C^2\}$ for some $V \subset \mathcal{D}_o$ and $F \subset \mathcal{F}_o$.

Again, for a given $\mathcal{D}$, there exist only finitely many $\mathcal{D}$-admissible sets. One proves in the same way as in 2.3 that $\mathbf{F}(\overline{X})$ is $\mathcal{D}$-admissible. This ends the proof of Theorem 2.4.

2.5. Consider a $\mathbf{Q}$-factorial, projective variety X. Assume that G acts on X with an open orbit of complexity at most one. Let $\varphi : X \to X'$ be the contraction of an extremal ray R of $NE(X)$. We assume that φ is birational, and moreover an isomorphism in codimension at one; we denote by $\varphi^+ : X^+ \to X'$ the flip of φ (see 1.4). We call this flip *direct* (resp. *inverse*) if $K_X < 0$ (resp. $K_X > 0$) on $R \setminus \{0\}$.

Theorem. *Under the assumptions above, every sequence of direct flips is finite, and every sequence of inverse flips as well.*

Proof. By 2.2, there are only finitely many isomorphism classes of G-varieties which are obtained from X by a sequence of flips. This implies our statement, by using [KMM] Proposition 5.1.11 (3); see [B] 4.7 for details.

Corollary. *Let X be a $\mathbf{Q}$-factorial, projective G-variety of complexity at most one. Assume that the morphism $G \to X : g \to g \cdot x$ is dominant and separable for some $x \in X$. Then there exists a projective, $\mathbf{Q}$-factorial G-variety X' and a birational, G-equivariant map $\varphi : X - \to X'$ such that:*

(i) φ factors through inverse flips and divisorial contractions of positive extremal rays.

(ii) $-K_{X'}$ is semi-ample.

Moreover, there exists a projective, $\mathbf{Q}$-Gorenstein G-variety X'' and a birational G-morphism $\varphi' : X' \to X''$ such that $-K_{X''}$ is ample.

Proof. Observe that the contraction φ of a non-negative extremal ray of $NE(X)$ is always birational. Namely, let C be an irreducible curve in X such that $\varphi(C)$ is a point. We show that C does not meet the open G-orbit in X. Otherwise, we may choose $\xi_1, \ldots, \xi_d$ in $Lie(G)$, and $x \in C$, such that C is smooth at x, the orbit $G \cdot x$ is open in X, and that the vectors $\xi_1 \cdot x, \ldots, \xi_d \cdot x$ form a basis of the tangent space of X at x. Then $s := \xi_1 \wedge \cdots \wedge \xi_d$ is a global section of $-K_X$, which does not vanish at x. Therefore, we have: $(-K_X \cdot C) \geq 0$. But $(K_X \cdot C) \geq 0$ by assumption. So $(K_X \cdot C) = 0$ and s has no zero on C. It follows that C is contained in the open G-orbit. Then the isotropy group G_x is infinite, and its connected component G_x^0 is not normal in G (otherwise $G \cdot x$ is affine; but $G \cdot x$ contains a projective curve). Now we can choose $\xi_1 \ldots, \xi_d$ as before, such that $\xi_1 \in Lie(G_y)$ for some $y \in C$. Then s vanishes at y, a contradiction.

Now the proof of the corollary is the same as [B] 4.7 Corollaire.

Remark. The separability assumption cannot be removed in the corollary. Namely, in every characteristic $p > 0$, we construct an example of a projective homogeneous variety X of complexity zero such that K_X and $-K_X$ are not semi-ample. Consider the group $G := \mathrm{SL}(3, k)$. The Frobenius endomorphism F of k extends to an endomorphism of G. We denote by V the G-module k^3, by V^* the dual G-module, and by $\mathbf{P}(V)$, $\mathbf{P}(V^*)$ the associated projective spaces. We let G act on $V \times V^*$ by $g \cdot (v, f) = ((F^2 g) \cdot v, g \cdot f)$. This defines a G-action on $\mathbf{P} := \mathbf{P}(V) \times \mathbf{P}(V^*)$. Clearly, B has a unique fixed point x in $\mathbf{P}$ and its isotropy group G_x is exactly B. Therefore, G has a unique closed orbit $X = G \cdot x$ in $\mathbf{P}$ and the map $G/B \to X$ is bijective. In particular, X is nonsingular, of complexity zero.

For any non-zero integer n, we claim that nK_X has no global section. Namely, X is a hypersurface in $\mathbf{P} \simeq \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, p^2)$ (the homogeneous equation of X is $(F^2 f)(v) = 0$). Therefore, we have $\omega_X = (\mathcal{O}_{\mathbf{P}^2}(-2) \otimes \mathcal{O}_{\mathbf{P}^2}(p^2 - 3))|_X$. Moreover, for any $v \in V$ with coordinates in the prime field, the image in $\mathbf{P}$ of the set $v \times (v = 0) \subset V \times V^*$ is a curve $C_v \subset X$, and $(K_X \cdot C_v) = p^2 - 3 > 0$. Similarly, for any $f \in V^*$ with coordinates in the prime field, we have a curve $C_f \subset X$ with $(K_X \cdot C_f) < 0$. This implies our claim.

Analogous considerations hold more generally for quotients of semisimple groups by non-reduced parabolic subgroup-schemes; see [W].

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