CLASSIFICATION OF SMOOTH AFFINE SPHERICAL VARIETIES

FRIEDRICH KNOP AND BART VAN STEIRTEGHEM

Abstract. Let G be a complex reductive group. A normal G-variety X is called spherical if a Borel subgroup of G has a dense orbit in X. Of particular interest are spherical varieties which are smooth and affine since they form local models for multiplicity free Hamiltonian K-manifolds, K a maximal compact subgroup of G. In this paper, we classify all smooth affine spherical varieties up to coverings, central tori, and $\mathbb{C}^\times$-fibrations.

1. Introduction

Let G be a connected reductive group (over $\mathbb{C}$). A normal G-variety X is called *spherical* if a Borel subgroup B of G has a dense orbit in it. When X is affine, this is equivalent to every simple G-module appearing at most once in $\mathbb{C}[X]$ ([VK]). In this article, we classify the smooth affine spherical varieties, up to coverings, central tori and $\mathbb{C}^\times$-fibrations (see Tables 1 through 5).

Our primary motivation are applications to Hamiltonian K-manifolds where $K \subseteq G$ is a maximal compact subgroup. These are symplectic K-manifolds which are equipped with a moment map $m : M \to \mathfrak{k}^*$. In [Sj], Sjamaar has shown that locally, a Hamiltonian K-manifold is isomorphic to a smooth affine G-variety. A Hamiltonian K-manifold is called *multiplicity free* if all symplectic reductions are zero-dimensional. Multiplicity free Hamiltonian manifolds have spherical varieties as local models (Brion, [Br2]). Therefore, our classification also yields a description of the local structure of multiplicity free Hamiltonian manifolds.

In [De], Delzant conjectured, based on results for commutative groups and for groups of rank 2, that a compact multiplicity free Hamiltonian K-manifold M is completely determined by its generic isotropy group and the image $m(M)$ of the moment map. The first named author was able to reduce this conjecture to a statement about smooth affine spherical varieties: such a variety is completely determined by the set of highest weights occurring in its ring of regular functions (Knop conjecture).

The present classification should constitute a major step toward verifying Knop's conjecture. Unfortunately, difficulties involving $\mathbb{C}^\times$-factors (see below) prevent us from immediately deducing the conjecture. The second step will be the subject of a forthcoming paper.

The starting point of our classification is a theorem of Luna, [Lu1], which implies that a general smooth affine spherical G-variety is a vector bundle $G \times^H V$ where $H \subseteq G$ is a reductive subgroup and V is an H-module.

In principle, it would suffice to classify the triples (G, H, V) but certain difficulties make the task infeasible. The problems come from $\mathbb{C}^\times$-factors in G or H and from H being non-connected (see Examples 2.3 through 2.6 for details). Therefore, in this paper, we just determine all triples $(\mathfrak{g}', \mathfrak{h}', V)$ where G and H are replaced by the semisimple part of their Lie algebras. In the process, we have lost some information. The original triple (G, H, V) can be recovered with additional combinatorial data but this point is not addressed in the present paper.

In our classification we build upon existing classifications of two "extreme" cases. The first is the case $V = 0$, i.e., $X = G/H$ is a homogeneous variety. Krämer [Kr] found a complete list of those when G is simple. His classification was later extended by Brion [Br3] and Mikityuk [Mi] to arbitrary G. The other extreme case is when $H = G$, i.e., when $X = V$ is a G-module. We call these *spherical G-modules*[1]. Another name for them is "multiplicity free spaces." Kac, [Kac], classified irreducible spherical G-modules and Brion, [Br1], the reducible spherical modules of simple groups. Later Benson-Ratcliff [BR] and Leahy [Le] independently completed the classification of spherical modules.

Now, for $G \times^H V$ to be spherical it is necessary that G/H be spherical for G and that V be spherical for H. The converse is not true. Therefore, we first derive a manageable criterion for $G \times^H V$ to be spherical. More precisely, for the pair (G, H) we define a certain subgroup L of H, the principal subgroup, with the property that $G \times^H V$ is spherical if and only if G/H is spherical and V is spherical for L (Theorem 5.3). We compute the principal subgroup for every variety in the Krämer-Brion-Mikityuk list. Combined with a couple of other useful "tricks" the classification turns out to be not too difficult.

While this work was in progress, Camus, [Ca], independently classified all smooth affine spherical varieties but only for groups of type A, i.e., when G is locally isomorphic to a product of $\mathbb{C}^\times$- and $SL(n)$-factors. His method is similar to ours except that we also use the first author's theory of actions on cotangent bundles, [Kn1, Kn3]. This simplified matters a lot, see e.g. the computation of the principal subalgebra, section 5.2, or the reflection "trick" of section 8.1.

Camus, on the other hand, overcame the above mentioned problems with $\mathbb{C}^\times$-factors by using a powerful theory of Luna [Lu2] and thereby deduced Knop's conjecture for groups of type A. Unfortunately, Luna's theory is only verified for groups of type A and certain other groups and Camus' method does not immediately carry over to the general case.

[1] This name collides with the notion of a "spherical representation", i.e., a representation with a K-fixed vector. These will never occur in this paper.

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Notation. All varieties we consider are defined over $\mathbb{C}$. G will always represent a connected reductive algebraic group. The Lie algebra of a group is always denoted by the corresponding fraktur letter. If $\mathfrak{g}$ is a Lie algebra, we put $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$. Fundamental weights ω_i are numbered as by Bourbaki [Bou]. A representation of a semisimple Lie algebra is described by its highest weight written multiplicatively. For example, $(\mathfrak{sl}(n) + \mathfrak{sp}(2k), \omega_1^2 + \omega_1\omega_1')$ represents the $\mathfrak{sl}(n)+\mathfrak{sp}(2k)$-module $S^2\mathbb{C}^n \oplus (\mathbb{C}^n \otimes \mathbb{C}^{2k})$. We will sometimes denote a $\mathfrak{g}$-module V by $(\mathfrak{g}, V)$ to stress which Lie algebra it is a representation of. To distinguish the n-dimensional abelian Lie algebra from an n-dimensional representation we denote it by $\mathfrak{t}^n$. If V is a vector space, then V^* is its dual.

2. Primitive spherical triples

Our starting point is the following well-known application of Luna's Slice Theorem [Lu1, Corollaire 2, p.98]:

Theorem 2.1. *Let X be a smooth affine G-variety with $\mathbb{C}[X]^G = \mathbb{C}$. Then $X \simeq G \times^H V$ where H is a reductive subgroup of G and V is an H-module.*

This applies to spherical varieties in the following way:

Corollary 2.2. *Let X be a smooth affine spherical G-variety. Then $X \simeq G \times^H V$ where H is a reductive subgroup of G such that G/H is spherical and V is a spherical H-module.*

Proof. Let $B \subseteq G$ be a Borel subgroup. Then B has an open orbit in X which implies $\mathbb{C}[X]^G = \mathbb{C}$. Moreover, this open orbit projects to an open orbit in G/H showing that G/H is spherical. Replacing B by a conjugate if necessary we may assume that $eH \in G/H$ is in this open orbit. Then $(B \cap H)^\circ$ and, a fortiori, a Borel subgroup of H has an open orbit in V. □

In the following, we determine (more or less) all triples (G, H, V) such that $G \times^H V$ is spherical. The spherical homogeneous spaces G/H correspond to the triples $(G, H, 0)$ and have been classified by Krämer [Kr], Brion [Br3], and Mikityuk [Mi], while the spherical H-modules correspond to (H, H, V) and are known thanks to Kac [Kac], Benson-Ratcliff [BR], and Leahy [Le]. The main task of the present paper is therefore to determine all "mixed" cases with $H \subsetneq G$ and $V \neq 0$.

Next, we present some simple reduction steps. Given any two triples (G_1, H_1, V_1) and (G_2, H_2, V_2), we can form their product $(G_1 \times G_2, H_1 \times$

$H_2, V_1 \oplus V_2$) to obtain a third. Our classification is therefore one of 'indecomposable' triples. As the following examples show, three other, more subtle phenomena have to be controlled in order to make the problem manageable.

Example 2.3 (H disconnected)**.** Consider the group $G(n) := SL(n)$ and its subgroup $H(n) := SL(n-1)$. If $n \geq 3$, then the homogeneous space $G(n)/H(n)$ is spherical. Now put $G := G(n_1) \times \ldots \times G(n_s)$ with integers $n_1 \geq \ldots \geq n_s \geq 3$ and $\bar{H} := H(n_1) \times \ldots \times H(n_s)$. Then $G/\bar{H}$ is also a spherical variety which is clearly highly decomposable. Now let $N \subseteq G$ be the normalizer of $\bar{H}$ in G (it is isomorphic to $GL(n_1-1) \times \ldots \times GL(n_s-1)$). Then G/H is spherical too, where H is any subgroup with $\bar{H} \subseteq H \subseteq N$. These subgroups are in one-to-one correspondence to subgroups A of $N/\bar{H} \simeq (\mathbb{C}^\times)^s$. Now choose for A a "very diagonal" finite subgroup, e.g., the group of d-th roots of unity, $d \geq 2$, embedded diagonally into $(\mathbb{C}^\times)^s$. Then the corresponding H is disconnected with $H^\circ = \bar{H}$ making G/H is indecomposable.

Using that G/H is spherical if and only if G/H° is we bypass this problem by considering triples $(\mathfrak{g}, \mathfrak{h}, V)$ where we have replaced the groups by their Lie algebras.

Example 2.4 ($\mathbb{C}^\times$ factors in H)**.** We keep the notation of the previous example. Now we take A to be a "very diagonal" connected subgroup, e.g., $\mathbb{C}^\times$ embedded diagonally into $(\mathbb{C}^\times)^s$. Then again $G/\bar{H}$ is indecomposable.

Example 2.5 ($\mathbb{C}^\times$ factors in G)**.** Let $G_0 := SL(n_1) \times \ldots \times SL(n_s)$ and $G_1 := GL(n_1) \times \ldots \times GL(n_s)$ with $n_1 \geq \ldots \geq n_s \geq 2$ acting on $V := \mathbb{C}^{n_1} \oplus \ldots \oplus \mathbb{C}^{n_s}$. Then V is a highly decomposable spherical module for both G_0 and G_1. Now let G be any intermediate connected group, determined by a connected subgroup of $G_1/G_0 \simeq (\mathbb{C}^\times)^s$. Then V will in general be an indecomposable spherical variety.

The obvious answer to the problems raised by the last two examples is to associate to the spherical variety $G \times^H V$ the triple $(\mathfrak{g}', \mathfrak{h}', V)$ where we replaced the Lie algebras by their commutator subalgebras. It contains slightly less information. Moreover, we run into another problem:

Example 2.6 (Missing $\mathbb{C}^\times$ factors)**.** (1) Let $T \simeq \mathbb{C}^\times$ be a maximal torus in $SL(2)$. To the spherical variety $SL(2)/T$ corresponds the triple $(\mathfrak{sl}(2), 0, 0)$, because $\mathfrak{t}' = 0$, but $SL(2)/e$ is obviously not spherical.
(2) $V = \mathbb{C}^n$ is spherical as a $\mathbb{C}^\times \times SO(n)$-variety but not as an $SO(n)$-variety. Nevertheless, the associated triple is $(\mathfrak{so}(n), \mathfrak{so}(n), \omega_1)$ in both cases.

Taking these issues into account, we define the actual objects of our classification.

Definition 2.7. *(i)* Let $\mathfrak{h} \subseteq \mathfrak{g}$ be semisimple Lie algebras and let V be a representation of $\mathfrak{h}$. For $\mathfrak{s}$, a Cartan subalgebra of the centralizer

$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$, put $\bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{s}$, a maximal central extension of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mathfrak{z}$ be a Cartan subalgebra of $\mathfrak{gl}(V)^{\mathfrak{h}}$ (the centralizer of $\mathfrak{h}$ in $\mathfrak{gl}(V)$). We call $(\mathfrak{g}, \mathfrak{h}, V)$ a *spherical triple* if there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a vector $v \in V$ such that
  (a) $\mathfrak{b} + \bar{\mathfrak{h}} = \mathfrak{g}$ and
  (b) $[(\mathfrak{b} \cap \bar{\mathfrak{h}}) + \mathfrak{z}]v = V$ where $\mathfrak{s}$ acts via any homomorphism $\mathfrak{s} \to \mathfrak{z}$ on V.

(ii) Two triples $(\mathfrak{g}_i, \mathfrak{h}_i, V_i)$, $i = 1, 2$, are *isomorphic* if there exist linear bijections $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\beta : V_1 \to V_2$ such that
  (a) α is a Lie algebra homomorphism;
  (b) $\alpha(\mathfrak{h}_1) = \mathfrak{h}_2$;
  (c) $\beta(\xi v) = \alpha(\xi)\beta(v)$ for all $\xi \in \mathfrak{h}_1$ and $v \in V_1$.

(iii) Triples of the form $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2, V_1 \oplus V_2)$ with $(\mathfrak{g}_i, \mathfrak{h}_i, V_i) \neq (0, 0, 0)$ are called *decomposable*.

(iv) Triples of the form $(\mathfrak{k}, \mathfrak{k}, 0)$ and $(0, 0, V)$ are said to be *trivial*.

(v) A pair $(\mathfrak{g}, \mathfrak{h})$ of semisimple Lie algebras is called *spherical* if $(\mathfrak{g}, \mathfrak{h}, 0)$ is a spherical triple.

(vi) A spherical triple (or pair) is *primitive* if it is non-trivial and indecomposable.

Remarks 2.8. *(i)* Recall from the introduction that a $\mathfrak{g}$-module V is called spherical if there is a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ and $v \in V$ such that $\mathfrak{b}v = V$. We also say that V is spherical for $\mathfrak{g}$. This condition is stronger than "the triple $(\mathfrak{g}, \mathfrak{g}, V)$ is spherical". More precisely, the latter is equivalent to V being spherical for $\mathfrak{g} + \mathfrak{z}$.

(ii) If $V = V_1 \oplus \ldots \oplus V_s$ is a decomposition of the $\mathfrak{h}$-module V in irreducible representations, then we can take $\mathfrak{z} = \mathfrak{t}^s$ with every $\mathbb{C}$-factor acting as scalars on the corresponding factor V_i.

(iii) The definition of a spherical triple is independent of the choice of the Cartan subalgebras $\mathfrak{s}$ and $\mathfrak{z}$.

(iv) For a spherical triple it is very rare that $\mathfrak{s} \neq \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ or $\mathfrak{z} \neq \mathfrak{gl}(V)^{\mathfrak{h}}$ but is does happen, e.g., for $(\mathfrak{sl}(2), 0, 0)$ and $(\mathfrak{sl}(n), \mathfrak{sl}(n), 2\omega_1)$, respectively.

(v) If the first component of two isomorphic triples $(\mathfrak{g}_i, \mathfrak{h}_i, V_i)$ is the same, say, $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$ then it should be kept in mind that α might be an outer isomorphism. For example, the triples $(\mathfrak{h}, \mathfrak{h}, V)$ and $(\mathfrak{h}, \mathfrak{h}, V^*)$ are always isomorphic in our sense even though V and V^* might be different as representations of $\mathfrak{h}$. Other examples are the pairs $(\mathfrak{so}(8), \mathfrak{so}(7))$ and $(\mathfrak{so}(8), \mathfrak{spin}(7))$ which are isomorphic due to triality.

(vi) From the semisimplicity of $\mathfrak{g}$ and $\mathfrak{h}$ (and V) it follows that the decomposition of a spherical triple $(\mathfrak{g}, \mathfrak{h}, V)$ into indecomposable triples is unique up to isomorphism.

(*vii*) A trivial summand of the form $(\mathfrak{k}, \mathfrak{k}, 0)$ corresponds to a factor K of G which acts trivially on X. A trivial summand $(0, 0, \mathbb{C})$ arises if X fibers as a line bundle over a smaller space.

The next theorem justifies our definition:

Theorem 2.9. *If $G \times^H V$ is a smooth affine spherical variety then $(\mathfrak{g}', \mathfrak{h}', V)$ is a spherical triple. Moreover, it follows from the classification that every spherical triple arises this way.*

Proof. Choose a Borel subgroup $B \subseteq G$ and $v \in V$ such that the B-orbit of $[1, v] \in G \times^H V$ is dense. This is equivalent to the density of BH in G and of $(B \cap H)v$ in V. In terms of Lie algebras we get $\mathfrak{b} + \mathfrak{h} = \mathfrak{g}$ and $(\mathfrak{b} \cap \mathfrak{h})v = V$.

Now let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{a}$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}$. Since the conditions of Definition 2.7 are independent on the choice of $\mathfrak{s}$ and $\mathfrak{z}$ we may arrange that the image of $\mathfrak{d}$ in $\mathfrak{g}'$ and $\mathfrak{gl}(V)$ is contained in $\mathfrak{s}$ and $\mathfrak{z}$, respectively. Let, in abuse of notation, $\mathfrak{b}' = \mathfrak{b} \cap \mathfrak{g}'$, a Borel subalgebra of $\mathfrak{g}'$. Then the sphericality of $(\mathfrak{g}', \mathfrak{h}', V)$ follows from the following two computations:

$$\mathfrak{g}' \oplus \mathfrak{a} = (\mathfrak{b}' \oplus \mathfrak{a}) + (\mathfrak{h}' + \mathfrak{d}) \subseteq (\mathfrak{b}' \oplus \mathfrak{a}) + (\mathfrak{h}' + \mathfrak{s} + \mathfrak{a}) = (\mathfrak{b}' + \mathfrak{h}' + \mathfrak{s}) \oplus \mathfrak{a}$$

and

$$V = ((\mathfrak{b}' \oplus \mathfrak{a}) \cap (\mathfrak{h}' + \mathfrak{d}))v \subseteq (\mathfrak{b}' \cap (\mathfrak{h}' + \mathfrak{s}) \oplus \mathfrak{z})v.$$

The last inclusion is seen as follows: let $h + d \in \mathfrak{h}' + \mathfrak{d}$ with $h + d \in \mathfrak{b}' \oplus \mathfrak{a}$. Then $(h + d)v \in (h + \mathfrak{z})v = (h + s + \mathfrak{z})v$ where $s \in \mathfrak{s}$ is the projection of d into $\mathfrak{g}'$. Then $h + s \in \mathfrak{b}' \cap (\mathfrak{h}' + \mathfrak{s})$ and the inclusion follows.

For the second part, let G' be the simply connected group with Lie algebra $\mathfrak{g}'$, and let $H' \subseteq G'$ be the connected subgroup with Lie algebra $\mathfrak{h}'$. Furthermore, let S and Z be maximal tori in the centralizers $C_{G'}(H')$ and $GL(V)^{H'}$, respectively.

Next we need that the action of $\mathfrak{h}'$ on V integrates to an action of H' on V. Since H' is, in general, not simply connected this is a non-trivial condition. It follows a posteriori, by inspection of Table 2 and the inference rules (Table 3).

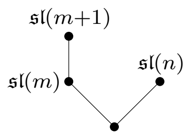
Granted this, we define $G := G' \times S \times Z$ and $H := H' \times S \times Z$. Here $S \subseteq H$ embeds diagonally into $G' \times S$ and acts trivially on V. We claim that $G \times^H V$ is spherical. Indeed, $\mathfrak{b} + \mathfrak{h} = (\mathfrak{b}' \oplus \mathfrak{s} \oplus \mathfrak{z}) + (\mathfrak{h}' \oplus \Delta\mathfrak{s} \oplus \mathfrak{z}) = (\mathfrak{b}' + \mathfrak{h}' + \mathfrak{s}) \oplus \mathfrak{s} \oplus \mathfrak{z} = \mathfrak{g}$ and $(\mathfrak{b} \cap \mathfrak{h})v = ((\mathfrak{b}' \cap (\mathfrak{h}' + \mathfrak{s})) \oplus \mathfrak{z})v = V$, since $\mathfrak{s}$ acts through $\mathfrak{z}$. ☐

3. Diagrams

To get a hold on the combinatorics involved we generalize the notation of Mikityuk [Mi] and represent a triple by a three-layered graph Γ as follows: Let $\mathfrak{g} = \mathfrak{g}_1 + \ldots + \mathfrak{g}_r$, $\mathfrak{h} = \mathfrak{h}_1 + \ldots \mathfrak{h}_s$, $V = V_1 \oplus \ldots \oplus V_t$ be the decompositions into simple factors. The vertices of the graph are all $\mathfrak{h}_j$, all V_k, and those $\mathfrak{g}_i$ which are not contained in $\mathfrak{h}$. There is an edge between $\mathfrak{h}_j$ and $\mathfrak{g}_i$ if $\mathfrak{h}_j \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_i$ is non-zero and an edge between V_k and $\mathfrak{h}_j$ if V_k is a non-trivial $\mathfrak{h}_j$-module. There is no edge between V_k and $\mathfrak{g}_i$.

In practice, we write the graph in three rows with the $\mathfrak{g}_i$-vertices, $\mathfrak{h}_j$-vertices, and V_k-vertices in the first, second, and third row respectively. In principle, all edges should be labeled to say how $\mathfrak{h}_j$ is embedded into $\mathfrak{g}_i$ or how $\mathfrak{h}_j$ acts on V_k. In most cases these embeddings and actions are the "natural" ones and we then omit the labels. Note that the $\mathfrak{g}_i$ which are contained in $\mathfrak{h}$ can be recovered from Γ: they correspond to those vertices $\mathfrak{h}_j$ which are not connected to any $\mathfrak{g}_i$.

Example 3.1. The graph



represents the triple $(\mathfrak{sl}(m+1)+\mathfrak{sl}(n), \mathfrak{sl}(m)+\mathfrak{sl}(n), \omega_1\omega_1')$.

4. The classification

Theorem 4.1. *(1) Every primitive spherical triple* $(\mathfrak{g}, \mathfrak{h}, V)$ *with* $V \neq 0$ *is contained in Table 1 or 2 or can be obtained from an item in these tables by applying the inference rules (Table 3), possibly several times.*

(2) If $(\mathfrak{g}, \mathfrak{h}, 0)$ *is a primitive spherical triple then* $(\mathfrak{g}, \mathfrak{h})$ *is contained in either Table 4 or Table 5.*

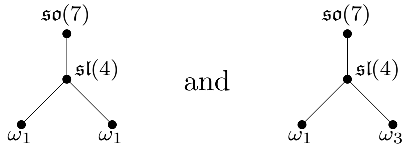
Moreover, every triple in these tables is spherical.

Part (2) of the theorem is just added for completeness since it restates the classifications of Krämer, Brion, and Mikityuk. Table 1 restates the classification of spherical modules by Kac, Benson-Ratcliff, and Leahy. New are Tables 2 and 3.

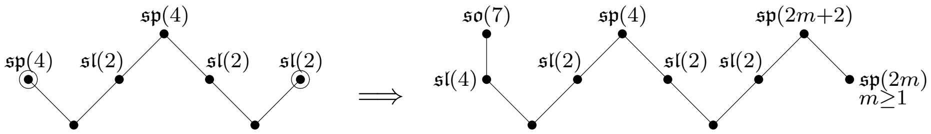
Explanation of the inference rules: They were introduced to keep the table at a moderate size. Expanding all items using these rules would result in 30+ more cases. A circled vertex ⦿ should remind the reader that an application of an inference rule is possible at that vertex.

4.1. The first rule means that every $(\mathfrak{sl}(2), \mathfrak{sl}(2))$-summand of $(\mathfrak{g}, \mathfrak{h})$ can be replaced by $(\mathfrak{sp}(2m+2), \mathfrak{sl}(2)+\mathfrak{sp}(2m))$ with $m \geq 1$.

4.2. The second rule states that if $(\mathfrak{g}, \mathfrak{h})$ contains the summand $(\mathfrak{sp}(4), \mathfrak{sp}(4))$ and, as an $\mathfrak{sp}(4)$-module, V contains only the trivial or the defining representation then that summand can be replaced by $(\mathfrak{so}(7), \mathfrak{sl}(4))$ (with embedding $\mathfrak{sl}(4) \simeq \mathfrak{so}(6) \hookrightarrow \mathfrak{so}(7)$). Moreover, the representation $(\mathfrak{sp}(4), \omega_1)$ can be replaced by either extension $(\mathfrak{sl}(4), \omega_1)$ or $(\mathfrak{sl}(4), \omega_3)$. Since there is an element of $\mathfrak{so}(7)$ which induces the outer automorphism of $\mathfrak{sl}(4)$ this choice is only relevant if $\mathfrak{sp}(4)$ acts non-trivially on more than one component of V. There is exactly one such case and it yields



Example 4.2. Applying both rules to the same diagram:



Another measure we took to cut down on the length of the table was allowing borderline cases: whenever an $\mathfrak{h}_j$ is 0, i.e., $\mathfrak{h}_j = \mathfrak{sl}(n)$, $n \leq 1$ or $\mathfrak{sp}(2n)$, $n \leq 0$ then that vertex and all adjacent edges are to be omitted. For that reason, Table 5 contains at first sight fewer entries than the tables of Brion and Mikityuk. Incidentally, the table of Krämer contains a couple of redundancies which we removed, e.g., $(\mathfrak{so}(8), \mathfrak{sp}(4) + \mathfrak{sl}(2))$ is, using triality, just $(\mathfrak{so}(8), \mathfrak{so}(5) + \mathfrak{so}(3))$.

5. Tools

5.1. Criterion for sphericality. Our main tool is a manageable criterion for a triple to be spherical. It will replace the 'fiber condition' in Definition 2.7 with a sphericality condition on a representation of a *reductive* Lie algebra. We are going to derive the criterion from [Kn1, Kn3] but we could also have used [Pa1, Pa2].

Let X be a G-variety. Then $L \subseteq G$ is called a *generic isotropy subgroup* if L is conjugate to G_x for all x in a non-empty open subset of X. If one exists, it is unique up to conjugacy. We are going to use this concept only in case X is a vector space. In that case, the existence of a generic isotropy subgroup is guaranteed by theorems of Richardson [Ri] and Luna [Lu1]. The Lie algebra of L is called a *generic isotropy subalgebra.*

Definition 5.1. Let $(\mathfrak{g}, \mathfrak{h})$ be a spherical pair and $\bar{\mathfrak{h}} = \mathfrak{h} + \mathfrak{s} \subseteq \mathfrak{g}$ a maximal central extension. Let $\bar{\mathfrak{l}}$ be a generic isotropy subalgebra of $\bar{\mathfrak{h}}$ acting on $\bar{\mathfrak{h}}^\perp$, the orthogonal complement of $\bar{\mathfrak{h}}$ in $\mathfrak{g}$. Then the image $\mathfrak{l}$ of $\bar{\mathfrak{l}}$ under the projection $\mathfrak{h} + \mathfrak{s} \twoheadrightarrow \mathfrak{h}$ is called a *principal subalgebra* of $(\mathfrak{g}, \mathfrak{h})$.

It is known that $\bar{\mathfrak{l}}$, and hence $\mathfrak{l}$, is always reductive ([Kn1, Korollar 8.2], [Pa1]). The following lemma establishes the connection of $\bar{\mathfrak{l}}$ with Borel subgroups. It is also contained in [Pa1].

Lemma 5.2. *Let H be a reductive subgroup of G such that $BH \subseteq G$ is dense. Then $B \cap H$ is a Borel subgroup of a generic isotropy group for H acting on $\mathfrak{h}^\perp$.*

Proof. Let U be the open subset of $\mathfrak{h}^\perp$ such that for every $u \in U$, H_u is H-conjugate to a generic isotropy group L and put $x_o := eH \in G/H$. From [Kn3, Theorem 3.2] we obtain a B-invariant section $\sigma : Bx_o \to T^*_{G/H} = G \times^H \mathfrak{h}^\perp$ which intersects $G \times^H U$ non-trivially. This implies $v \in U$ where $\sigma(x_o) = [e, v]$. Hence

$$B \cap H = B_{x_o} \subseteq G_{\sigma(x_o)} = H_v = L.$$

On the other hand, Corollaries 2.4 and 8.2 in [Kn1] imply that $B \cap H$ is G-conjugate to a Borel subgroup of L. Therefore, it is a Borel subgroup of L. □

The criterion is now:

Theorem 5.3. *The triple* $(\mathfrak{g}, \mathfrak{h}, V)$ *is spherical if and only if* $(\mathfrak{g}, \mathfrak{h})$ *is a spherical pair and* V *is a spherical* $\mathfrak{l} + \mathfrak{z}$*-module. Here* $\mathfrak{l}$ *is a principal subalgebra of* $(\mathfrak{g}, \mathfrak{h})$ *and* $\mathfrak{z}$ *is a Cartan subalgebra of* $\mathfrak{gl}(V)^{\mathfrak{h}}$*.*

Proof. Assume $(\mathfrak{g}, \mathfrak{h})$ is a spherical pair. Lemma 5.2 implies that $\mathfrak{b} \cap (\mathfrak{h} + \mathfrak{s})$ is a Borel subalgebra of $\bar{\mathfrak{l}} \subseteq \mathfrak{h} + \mathfrak{s}$. Thus, $(\mathfrak{g}, \mathfrak{h}, V)$ is spherical if and only if V is a spherical $\bar{\mathfrak{l}} + \mathfrak{z}$-module. The image of $\mathfrak{b} \cap (\mathfrak{h} + \mathfrak{s})$ in $\mathfrak{h}$ is a Borel subalgebra of $\mathfrak{l}$. Since $\mathfrak{s}$ acts on V through $\mathfrak{z}$, we can replace $\bar{\mathfrak{l}}$ by $\mathfrak{l}$. □

Observe that "V is a spherical $\mathfrak{l} + \mathfrak{z}$-module" is a stronger condition than "$(\mathfrak{l}', \mathfrak{l}', V)$ is a spherical triple", since $\mathfrak{h}$-irreducible modules may not be $\mathfrak{l}$-irreducible. In other words, in the latter statement, $\mathfrak{z}$ could be bigger. To deal with this kind of "$\mathbb{C}^\times$-deficiency" one can use the following

Lemma 5.4. *Let* $(\mathfrak{h}, V)$ *be a spherical module and* $\mathfrak{z} \subseteq \mathfrak{gl}(V)^{\mathfrak{h}}$ *a Cartan subalgebra. Then there is a (unique) subspace* $\mathfrak{c} \subseteq \mathfrak{z}$ *such that for every subspace* $\mathfrak{z}_0 \subseteq \mathfrak{z}$*:*

$$V \text{ is spherical for } \mathfrak{h} + \mathfrak{z}_0 \qquad \Leftrightarrow \qquad \mathfrak{z}_0 + \mathfrak{c} = \mathfrak{z}.$$

For a proof see [Kn4] Thm. 5.1. In the notation of that paper we have $\mathfrak{c} = (\mathfrak{a}^* \cap \mathfrak{z}^*)^\perp$. For the convenience of the reader we list $\mathfrak{c}$ for the cases we are going to use in the sequel: we have $\mathfrak{c} = 0$ for

$$(\mathfrak{h}, V) = (\mathfrak{so}(n), \omega_1), (\mathfrak{so}(8), \omega_3 + \omega_4), (\mathfrak{sp}(2m) + \mathfrak{sl}(2) + \mathfrak{sp}(2n), \omega_1\omega_1' + \omega_1'\omega_1'').$$

In other words, for these pairs, V will not remain spherical if $\mathfrak{z}$ is replaced by any proper subspace $\mathfrak{z}_0$. In the other extreme, we have $\mathfrak{c} = \mathfrak{z}$ for $(\mathfrak{h}, V) = (\mathfrak{sp}(2n), \omega_1)$. Finally, we have the following intermediate cases:

$\mathfrak{c} = \mathbb{C}(1,1)$ for $(\mathfrak{h}, V) = (\mathfrak{sl}(n), \omega_1 + \omega_1), n \geq 2$.

$\mathfrak{c} = \mathbb{C}(1,-1)$ for $(\mathfrak{h}, V) = (\mathfrak{sl}(n), \omega_1 + \omega_{n-1}), n \geq 3$.

$\mathfrak{c} \subseteq \mathbb{C}(1,1)$ for $(\mathfrak{h}, V) = (\mathfrak{sl}(2) + \mathfrak{sl}(n), \omega_1\omega_1' + \omega_1'), n \geq 2$.

Note that in the first and the third case the scalar $\mathfrak{t}^1$-action on V does *not* make V spherical for $\mathfrak{h} \oplus \mathfrak{t}^1$.

5.2. Calculating principal subalgebras. The next step is to determine the principal subalgebras.

Proposition 5.5. *The principal subalgebras of all primitive spherical pairs are listed in Tables 4 and 5.*

The verification of the data is quite standard and is left to the reader. In the calculations, Elashvili's tables in [El1] and [El2] are very useful, especially, if $\mathfrak{s} = 0$. In all other cases we have $\mathfrak{s} = \mathfrak{t}^1$ (by inspection). In the tables, these cases are indicated by a "$*$". For them, the following observation proves useful since Elashvili's tables only contain generic isotropy algebras for representations of semisimple Lie algebras.

Lemma 5.6. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a primitive spherical pair with* $\mathfrak{s} = \mathfrak{t}^1$*, let* $\mathfrak{l}$ *be a principal subalgebra, and let* $\mathfrak{l}_*$ *be a generic isotropy subalgebra of* $\mathfrak{h}$ *acting on* $\mathfrak{h}^\perp$*. Then the following statements are equivalent:*

(*i*) $\mathfrak{l} \simeq \mathfrak{l}_* \oplus \mathfrak{t}^1$.
(*ii*) *G/H is spherical.*
(*iii*) *Every* $\mathfrak{h}$*-invariant on* $\bar{\mathfrak{h}}^\perp$ *is* $\bar{\mathfrak{h}}$*-invariant.*

If these conditions do not hold then $\mathfrak{l} = \mathfrak{l}_*$.

Proof. First observe that the map $\bar{\mathfrak{l}} \to \mathfrak{l}$ is bijective. Otherwise $\mathfrak{s} \subseteq \bar{\mathfrak{l}}$, since $\dim \mathfrak{s} = 1$, and $\mathfrak{s}$, being normal in $\mathfrak{h} + \mathfrak{s}$, would act trivially on $(\mathfrak{h} + \mathfrak{s})^\perp$. This implies that $\mathfrak{s}$ is in the center of $\mathfrak{g}$. Contradiction.

Let $\bar{H} = S \cdot H$ and consider the $\mathbb{C}^\times$-fibration $X := G/H \to \bar{X} := G/\bar{H}$. Let $U \subseteq G$ be a maximal unipotent subgroup. Then the dimension of a generic U-orbit in X and $\bar{X}$ is the same, which implies that $P_u(X) = P_u(\bar{X})$ in the notation of [Kn2]. Therefore, the sum of rank and complexity is one smaller for $\bar{X}$ than for X ([Kn2, Korollar 2.12]). Thus, X is spherical if and only if its complexity is zero if and only if $\mathrm{rk} X = \mathrm{rk} \bar{X} + 1$ if and only if a generic isotropy group in T^*_X is of codimension one in that of $T^*_{\bar{X}}$. This proves the equivalence of (*i*) and (*ii*). The equivalence of (*ii*) and (*iii*) follows from [Kn1, Satz 7.1]. If X is not spherical then $\mathrm{rk} X = \mathrm{rk} \bar{X}$ and $\mathfrak{l}_* = \bar{\mathfrak{l}}$. □

Remark 5.7. If $\mathfrak{g}$ is simple, then information on (*ii*) is part of Krämer's table. For (*iii*), the tables of Schwarz [Sch] are useful.

5.3. A reduction lemma. In certain cases, it is not even necessary to calculate the principal subalgebra.

Lemma 5.8. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a spherical pair such that* $\bar{\mathfrak{h}}^\perp = U \oplus U^*$ *with* U *an irreducible spherical* $\bar{\mathfrak{h}}$*-module. Let* $\tilde{\mathfrak{h}} \subseteq \tilde{\mathfrak{g}}$ *be another pair of semisimple Lie algebras and let* V *be an* $\mathfrak{h} + \tilde{\mathfrak{h}}$*-module. Consider the following statements:*

(*i*) *The triple* $(\mathfrak{g} + \tilde{\mathfrak{g}}, \mathfrak{h} + \tilde{\mathfrak{h}}, V)$ *is spherical.*
(*ii*) *The triple* $(\mathfrak{h} + \tilde{\mathfrak{g}}, \mathfrak{h} + \tilde{\mathfrak{h}}, U \oplus V)$ *is spherical.*

Then (i) implies (ii). The converse is true if the action of $\bar{\mathfrak{h}}$ *on* U *contains the scalars.*

Proof. Clearly, both conditions imply that $(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}})$ is a spherical pair. Let $\mathfrak{l}\subseteq\mathfrak{h}$ and $\tilde{\mathfrak{l}}\subseteq\tilde{\mathfrak{h}}$ be the principal subalgebras and $\mathfrak{z}\subseteq\mathfrak{gl}(V)^{\mathfrak{h}+\tilde{\mathfrak{h}}}$ a Cartan subalgebra. Then *(i)* holds if and only if V is spherical for $\mathfrak{l}+\tilde{\mathfrak{l}}+\mathfrak{z}$.

Let $\mathfrak{b}_0\subseteq\bar{\mathfrak{h}}$ be a Borel subalgebra. Since U is spherical for $\bar{\mathfrak{h}}$ there is $u\in U$ with $\mathfrak{b}_0u=U$. Let $\mathfrak{b}_1\subseteq\mathfrak{b}_0$ be its isotropy subalgebra. As in Lemma 5.2, $\mathfrak{b}_1$ is a Borel subalgebra of a generic isotropy subalgebra $\bar{\mathfrak{l}}$ for $\bar{\mathfrak{h}}$ acting on $T_U^*=U\oplus U^*=\bar{\mathfrak{h}}^\perp$. Thus, *(i)* holds if and only if $U\oplus V$ is spherical for $\bar{\mathfrak{h}}+\tilde{\mathfrak{l}}+\mathfrak{z}$. This implies that $U\oplus V$ is spherical for $\mathfrak{h}+\tilde{\mathfrak{l}}+\mathfrak{t}^1+\mathfrak{z}$, i.e., *(ii)*. Moreover, the last implication is an equivalence if $\mathfrak{t}^1\subseteq\bar{\mathfrak{h}}$. □

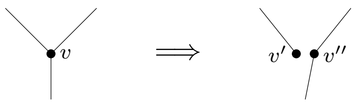
5.4. The cutting lemma. Clearly, a triple is trivial if its graph consists of only isolated vertices which are all in the $\mathfrak{h}$- and V-layers[2]. A triple is primitive if and only if it is non-trivial and its graph is connected.

The pair $(\mathfrak{g},\mathfrak{h})$ is called the *base* of the triple. Its graph Γ_b is obtained by removing the V-layer from Γ. A non-trivial component of $(\mathfrak{g},\mathfrak{h})$ is called a *base component.* It corresponds to a connected component of Γ_b which is not an isolated vertex in the $\mathfrak{h}$-level.

The triple $(\mathfrak{h},\mathfrak{h},V)$ is called the *fiber* of the triple. Its graph Γ_f is obtained by removing the $\mathfrak{g}$-layer from Γ. A component of the fiber is called a *fiber component.* It corresponds to a connected component of Γ_f.

The process of removing the top or bottom layer can be generalized. Let v be a vertex of Γ. A *cut of* Γ *in* v is a graph Γ' which has the same vertices as Γ except that v is replaced by two vertices v', v'' (in the same layer). Moreover, Γ' has the same edges as Γ with the property that each edge adjacent to v is in Γ' adjacent to either v' or v''.

Example 5.9.



We need the following

Lemma 5.10. *Let $U_1\neq0$, $U_2\neq0$, U_3 be representations of the reductive Lie algebra $\mathfrak{l}$. Assume that $(U_1\otimes U_2)\oplus U_3$ is a spherical $\mathfrak{l}+\mathfrak{t}^2$-module. Then $U_1\oplus U_2\oplus U_3$ is a spherical $\mathfrak{l}+\mathfrak{t}^3$-module.*

Proof. Consider the morphism

$$\pi:U_1\oplus U_2\oplus U_3\to(U_1\otimes U_2)\oplus U_3:(u_1,u_2,u_3)\mapsto(u_1\otimes u_2,u_3).$$

The image of π is a subvariety of a spherical variety, hence spherical. Moreover, π is generically a $\mathbb{C}^\times$-fibration. More precisely, it is the quotient by the $\mathbb{C}^\times$-action $t\cdot(u_1,u_2,u_3)=(tu_1,t^{-1}u_2,u_3)$, of which the infinitesimal action is contained in that of $\mathfrak{t}^3\subseteq\mathfrak{l}+\mathfrak{t}^3$ on $U_1\oplus U_2\oplus U_3$. This implies that $U_1\oplus U_2\oplus U_3$ is spherical as well. □

[2]Note that the non-trivial triple $(\mathfrak{sl}(2),0,0)$ is represented by an isolated vertex in the $\mathfrak{g}$-layer.

Now we can prove the cutting lemma:

Lemma 5.11. *Let Γ be the graph of a spherical triple and let Γ' be the graph obtained by cutting Γ in a vertex in the $\mathfrak{h}$-layer or in the V-layer. Then Γ' is also the graph of a spherical triple.*

Proof. 1. *Cut in the V-layer.* Let v be a vertex corresponding to an irreducible component V_k of $(\mathfrak{h}, V)$. The adjacent edges correspond to the simple factors of $\mathfrak{h}$ acting non-trivially on V_k. Thus there is a decomposition $V_k = U_1 \otimes U_2$ such that a simple factor h_j which is attached to v' or v'' (in Γ') acts only on U_1 or U_2, respectively. Write $V = V_k \oplus U_3$. Then the process of cutting amounts to replacing V by $U_1 \oplus U_2 \oplus U_3$. The assertion follows from Lemma 5.10.

2. *Cut in the $\mathfrak{h}$-layer.* Let v be a vertex corresponding to a simple factor $\mathfrak{h}_j$ of $\mathfrak{h}$. If both cut vertices v' and v'' are connetced to the $\mathfrak{g}$-layer then Γ' corresponds to the triple $(\mathfrak{g}, \mathfrak{h} \oplus \mathfrak{h}_j, V)$. Let $\bar{\mathfrak{l}}$ be a principal subalgebra of $(\mathfrak{g}, \mathfrak{h} \oplus \mathfrak{h}_j)$. Then $\mathfrak{l} \subseteq \bar{\mathfrak{l}}$ implies that the image of $\bar{\mathfrak{l}}$ in $\mathfrak{gl}(V)$ contains the image of $\mathfrak{l}$, which proves the assertion. If one of the verices v', v'' is not connected to the $\mathfrak{g}$-layer then Γ' represents the triple $(\mathfrak{g} \oplus \mathfrak{h}_j, \mathfrak{h} \oplus \mathfrak{h}_j, V)$. Its principal subalgebra is $\mathfrak{l} \oplus \mathfrak{h}_j$ and we argue as before. □

An immediate consequence is the erasing lemma:

Corollary 5.12. *Let $(\mathfrak{g}, \mathfrak{h}, V)$ be a spherical triple with graph Γ. Let $\tilde{\Gamma}$ be the graph obtained from Γ by erasing any number of edges between the $\mathfrak{h}$- and the V-layer. Then the triple $(\mathfrak{g}, \mathfrak{h}, \tilde{V})$ corresponding to $\tilde{\Gamma}$ is spherical, as well.*

Proof. Indeed, we can cut out every edge. □

6. The base components

In this section we start the classification by ruling out most of the primitive spherical pairs from being base components of a primitive triple with nonzero fiber.

Proposition 6.1. *Every primitive spherical pair which is a base component of a primitive spherical triple $(\mathfrak{g}, \mathfrak{h}, V)$ with $V \neq 0$ is contained in Table 6. The top two layers of the graph represent the component $(\mathfrak{g}_0, \mathfrak{h}_0)$ while the third layer indicates its principal subalgebra $\mathfrak{l}$ and its embedding into $\mathfrak{h}_0$.*

The rest of this subsection is devoted to the proof of this statement. Let $(\mathfrak{g}_0, \mathfrak{h}_0)$ be as in the proposition. Then the erasing Lemma (Corollary 5.12) implies that there is a primitive spherical triple of the form $(\mathfrak{g}, \mathfrak{h}, V)$ with $\mathfrak{g} = \mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{h}_0$ and where V is irreducible and exactly one simple factor $\mathfrak{h}_j$ of $\mathfrak{h}$ acts non-trivially on V.

Some simple algebras have more than one irreducible spherical module. The following lemma reduces the number of cases to check:

Lemma 6.2. *Let* $(\mathfrak{g}, \mathfrak{h}, V)$ *be a primitive spherical triple such that* V *is irreducible and exaclty one simple factor, say* $\mathfrak{h}_j$*, acts non-trivially on* V*.*

(i) Assume $\mathfrak{h}_j = \mathfrak{sl}(m)$*,* $m \geq 2$*, and* $V = \omega_1^2$*,* $V = \omega_{m-1}^2$*, or* $V = \omega_{m-1}$*. Then also* $(\mathfrak{g}, \mathfrak{h}, \omega_1)$ *is spherical.*

(ii) Assume $\mathfrak{h}_j = \mathfrak{sp}(4)$ *and* $V = \omega_2$*. Then also* $(\mathfrak{g}, \mathfrak{h}, \omega_1)$ *is spherical.*

Proof. Let $\mathfrak{l}$ be the principal subalgebra of $(\mathfrak{g}, \mathfrak{h})$ and $\tilde{\mathfrak{l}} = \mathfrak{l} \oplus \mathfrak{t}^1$.
(i) It is well known that a module is spherical if and only its dual is. Thus we may assume $V = \omega_1^2$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{gl}(m)$. Since $\dim \omega_1^2 = \dim \mathfrak{b}$, the image of $\tilde{\mathfrak{l}}$ in $\mathfrak{gl}(m)$ contains $\mathfrak{b}$, hence equals $\mathfrak{gl}(m)$.
(ii) Let $\mathfrak{l}_0$ be the image of $\mathfrak{l}$ in $\mathfrak{h}_j = \mathfrak{sp}(4)$. The Borel subalgebra of $\mathfrak{l}_0$ must have dimension at least $\dim V - 1 = 4$. This leaves only the possibilities $\mathfrak{l}_0 = \mathfrak{sp}(4)$ or $\mathfrak{l}_0 = \mathfrak{sl}(2) + \mathfrak{sl}(2)$. But in the latter case V is not spherical for $\mathfrak{l}_0 \oplus \mathfrak{t}^1$. □

Proof of Proposition 6.1. First, the items of Tables 4 and 5 marked by a "•" are members of Table 6. We examine the others.

Let $\mathfrak{l}$ be a principal subalgebra of $(\mathfrak{g}, \mathfrak{h})$ and $\tilde{\mathfrak{l}} = \mathfrak{l} \oplus \mathfrak{t}^1$. According to the list of irreducible spherical modules (in Table 1) and Lemma 6.2 the following cases for $(\mathfrak{h}_j, V)$ need to be checked:

- $(\mathfrak{h}_j, \omega_1)$ where $\mathfrak{h}_j$ is one of $\mathfrak{sl}(n)$, $\mathfrak{sp}(n)$, $\mathfrak{so}(n)$, G_2, or E_6;
- $(\mathfrak{sl}(n), \omega_2)$;
- the spin representation for $n = 7, 8, 9, 10$.

In the cases marked "b", the dimension of the Borel subalgebra of $\tilde{\mathfrak{l}}$ is smaller than $\dim V$. The same happens for "b'" but one has to take the image of $\mathfrak{l}$ in $\mathfrak{h}_j$ into account. In the cases marked "+", V contains two $\mathfrak{l}$-stable lines with the same character, hence cannot be $\tilde{\mathfrak{l}}$-spherical. The cases marked by an "x" will be checked separately:

$(\mathsf{E}_6, \mathsf{F}_4)$ is not in Table 6 because F_4 does not have non-trivial spherical modules.

For $(\mathsf{F}_4, \mathsf{B}_4)$, the principal subalgebra is $\mathfrak{l} = \mathsf{B}_3$, spin-embedded into $\mathfrak{so}(8) \subseteq \mathsf{B}_4$. (B_4, ω_4) is not spherical for $\tilde{\mathfrak{l}}$ because its Borel subalgebra is too small; and (B_4, ω_1) is not because the orbits of $\mathfrak{spin}(7)$ have codimension at least 2.

For $(\mathsf{E}_6, \mathsf{D}_5)$, the principal subalgebra is $\mathfrak{l} = \mathsf{A}_3 + \mathfrak{t}^1$ where $\mathsf{A}_3 \hookrightarrow \mathsf{B}_3 = \mathfrak{spin}(7) \hookrightarrow \mathsf{D}_4 \subseteq \mathsf{D}_5$. Therefore, the same reasoning as in the case $(\mathsf{F}_4, \mathsf{B}_4)$ applies.

The last x-case is $(\mathfrak{sl}(m{+}2) + \mathfrak{sp}(2n{+}2), \mathfrak{sl}(m) + \mathfrak{sl}(2) + \mathfrak{sp}(2n))$. The principal subalgebra is $\mathfrak{l} = \mathfrak{gl}(m{-}2) + \mathfrak{sp}(2n{-}2)$. We see that $(\mathfrak{gl}(m), \omega_1)$ and $(\mathfrak{sp}(2n), \omega_1)$ are not spherical for $\tilde{\mathfrak{l}}$ by "+", and that $(\mathfrak{gl}(m), \omega_2)$ is not by "b".

Finally, we come the two cases which are marked "∘". These form series which consist partially of possible base components for triples with nonzero fiber. More precisely,

(*i*) Let $m \geq n \geq 1$ and $m + n \geq 7$. If the spherical pair $(\mathfrak{so}(m + n), \mathfrak{so}(m) + \mathfrak{so}(n))$ is a base component then $n = 1$ or $n = 2$.
(*ii*) Let $\mathfrak{g}$ be a simple Lie algebra. If the spherical pair $(\mathfrak{g} + \mathfrak{g}, \mathfrak{g})$ is a base component then $\mathfrak{g} \simeq \mathfrak{sl}(n)$, $n \geq 2$.

Proof of (i). Suppose $n \geq 3$. Then $\mathfrak{l} = \mathfrak{so}(m-n) \subseteq \mathfrak{so}(m)$ implies that V is a spherical $\mathfrak{so}(m)$-module. Inspecting Table 1 we see that we have to check only for V the defining or the spin representation (for $m = 4$ and $m = 6$ we have to use Lemma 6.2(*i*)). Both acquire multiplicities when restricted to $\mathfrak{l} = \mathfrak{so}(m-n)$.

Proof of (ii). The principal subalgebra is a Cartan subalgebra of $\mathfrak{g}$. Thus, in order for $(\mathfrak{g} + \mathfrak{g}, \mathfrak{g}, V)$ to be spherical we must have $\dim V \leq \operatorname{rk}\mathfrak{g} + 1$. Inspecting Table 1, we see this is only possible for $\mathfrak{g} = \mathfrak{sl}(n)$.

This finishes the proof of Proposition 6.1. □

7. Reduced triples

In this section, we justify the inference rules.

Lemma 7.1. *Fix $m \geq 1$. Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be semisimple Lie algebras and let V be an $\mathfrak{sl}(2) + \mathfrak{h}_0$-module. Then the following are equivalent:*

(*i*) $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, V) = (\mathfrak{sl}(2) + \mathfrak{g}_0, \mathfrak{sl}(2) + \mathfrak{h}_0, V)$ *is a primitive spherical triple.*
(*ii*) $(\mathfrak{g}, \mathfrak{h}, V) = (\mathfrak{sp}(2m+2) + \mathfrak{g}_0, \mathfrak{sp}(2m) + \mathfrak{sl}(2) + \mathfrak{h}_0, V)$ *is a primitive spherical triple.*

Proof. The principal subalgebra of $(\mathfrak{sp}(2m+2), \mathfrak{sp}(2m) + \mathfrak{sl}(2))$ surjects onto $\mathfrak{sl}(2)$. The statement now follows from Theorem 5.3. □

Lemma 7.2. *Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be semisimple Lie algebras and let V be an $\mathfrak{so}(6) + \mathfrak{h}_0$-module. The following are equivalent:*

(*i*) $(\mathfrak{g}, \mathfrak{h}, V) = (\mathfrak{so}(7) + \mathfrak{g}_0, \mathfrak{so}(6) + \mathfrak{h}_0, V)$ *is a primitive spherical triple.*
(*ii*) $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, V) = (\mathfrak{so}(5) + \mathfrak{g}_0, \mathfrak{so}(5) + \mathfrak{h}_0, V)$ *is a primitive spherical triple such that $V|_{\mathfrak{so}(5)}$ contains only the trivial and the spin represntation.*

Moreover, V contains in this case, as an $\mathfrak{so}(6)$-module, at most the trivial and the two spin representations. All of these stay irreducible as $\mathfrak{so}(5)$-modules.

Proof. First assume (*i*). The principal subalgebra of $(\mathfrak{so}(7), \mathfrak{so}(6))$ is $\mathfrak{so}(5)$. Thus, Theorem 5.3 implies that $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, V) = (\mathfrak{so}(5) + \mathfrak{g}_0, \mathfrak{so}(5) + \mathfrak{h}_0, V)$ is spherical. Let U be a simple component of $V|_{\mathfrak{so}(6)}$. Then the erasing lemma (Corollary 5.12) implies that $(\mathfrak{so}(7), \mathfrak{so}(6), U)$ is spherical. Table 1 implies easily that U is either $\mathbb{C}$, $\mathbb{C}^4$, or $(\mathbb{C}^4)^*$. This proves (*ii*) and the last statement.

Now assume (*ii*). It is not possible to deduce the sphericality of $(\mathfrak{g}, \mathfrak{h}, V)$ directly from that of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}, V)$. A counterexample is $(\mathfrak{so}(7), \mathfrak{so}(6), \mathbb{C}^6)$. The problem is that the torus $\mathfrak{z}$ may be different. On the other hand, the assumption on $V|_{\mathfrak{so}(5)}$ implies easily that $V|_{\mathfrak{so}(6)}$ contains only the trivial and the spin representations. This means that $V|_{\mathfrak{so}(5)}$ and $V|_{\mathfrak{so}(6)}$ have the same number of irreducible components, i.e., $\mathfrak{z}$ does not change. □

Definition 7.3. A spherical triple $(\mathfrak{g}, \mathfrak{h}, V)$ is called *reduced* if it is primitive and

(*i*) $(\mathfrak{so}(7), \mathfrak{so}(6))$ is not a component of $(\mathfrak{g}, \mathfrak{h})$ and
(*ii*) $(\mathfrak{sp}(2m+2), \mathfrak{sp}(2m) + \mathfrak{sl}(2))$ is, for any $m \geq 1$, not a component of $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{sp}(2m)$ acts trivially on V.

Corollary 7.4. *All primitive spherical triples can be obtained from reduced spherical triples by (possibly repeated) application of the inference rules.*

Thus, Tables 1 and 2 constitute, in fact, a classification of reduced spherical triples.

8. Simple extensions

As one sees from the tables, the bulk of the new spherical triples is of the following type.

Definition 8.1. A reduced spherical triple $(\mathfrak{g}, \mathfrak{h}, V)$ is called a *simple extension (of* $(\mathfrak{g}_0, \mathfrak{h}_0)$*)* if

(*i*) it has exactly one base component $(\mathfrak{g}_0, \mathfrak{h}_0)$,
(*ii*) it has exactly one fiber component, and
(*iii*) the intersection of the base component with the fiber component is a single vertex $\mathfrak{h}_j$.

In other words, the diagram of $(\mathfrak{g}, \mathfrak{h}, V)$ is obtained by gluing a diagram of Table 6 to a diagram of Table 1 to one vertex in the $\mathfrak{h}$-level. In this section we prove:

Proposition 8.2. *All simple extensions are contained in Table 2.*

We start by determining all simple extensions which are glued at an $\mathfrak{sl}(2)$:

Lemma 8.3. *Let* $(\mathfrak{g}, \mathfrak{h}, V)$ *be a simple extension glued at* $\mathfrak{h}_j = \mathfrak{sl}(2)$. *Then* $(\mathfrak{g}, \mathfrak{h}, V)$ *is obtained by gluing any of the components*

— $(\mathfrak{sl}(n+2), \underline{\mathfrak{sl}(2)} + \mathfrak{sl}(n)), n \geq 1$
— $(\mathfrak{sp}(4) + \mathfrak{sp}(2n+2), \underline{\mathfrak{sl}(2)} + \mathfrak{sl}(2) + \mathfrak{sp}(2n)), n \geq 0$
— $(\mathfrak{sp}(2m+2) + \mathfrak{sp}(2n+2), \mathfrak{sp}(2m) + \underline{\mathfrak{sl}(2)} + \mathfrak{sp}(2n)), m, n \geq 0$

with any of the modules

— $(\underline{\mathfrak{sl}(2)} + \mathfrak{sl}(n), \omega_1\omega_1')$, $n \geq 1$
— $(\underline{\mathfrak{sl}(2)} + \mathfrak{sp}(2n), \omega_1\omega_1')$, $n \geq 2$

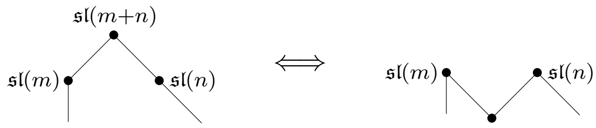
at an underlined factor. All of these are contained in Table 2.

Proof. Let $\mathfrak{l}_0$ be the image of $\mathfrak{l} \subseteq \mathfrak{h}$ in $\mathfrak{h}_j$. Then there are three possibilities: (1) $\mathfrak{l}_0 = 0$. In this case, one cannot glue $\mathfrak{h}_j$ to anything because, as an $\mathfrak{l}_0 + \mathfrak{z}$-module, V would have multiplicities. (2) $\mathfrak{l}_0 = \mathfrak{sl}_2$. Inspecting Table 6 this is precisely the case for $(\mathfrak{sp}(2m+2), \mathfrak{sl}(2)+\mathfrak{sp}(2m))$ which we dealt with in Lemma 7.1. (3) $\mathfrak{l}_0 = \mathfrak{t}^1$. These are precisely the cases listed in the lemma.

Now we claim that the listed modules are those which stay spherical when $\mathfrak{sl}(2)$ is replaced by $\mathfrak{t}^1$. In fact, a spherical module $(\mathfrak{sl}(2)+\mathfrak{h}_0, V)$ has this property if and only of the triple $(\mathfrak{sl}(3)+\mathfrak{h}_0, \mathfrak{sl}_2+\mathfrak{h}_0, V)$ is spherical. Now we can apply Lemma 5.8 with $U = \omega_1$. Thus, we see that the triple is spherical if and only if $(\mathfrak{sl}(2)+\mathfrak{h}_0, \mathfrak{sl}(2)+\mathfrak{h}_0, U \oplus V)$ is spherical. Using Table 1 we easily obtain the list of modules given. □

For the rest of the proof we are going through the list of possible base components (Table 6). The underlined factor is the one we want to glue, in case there is a choice.

8.1. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sl}(m+n), \underline{\mathfrak{sl}(m)} + \mathfrak{sl}(n))$, $m \geq 3$, $n \geq 1$**:** In this case Lemma 5.8 applies with $U = \omega_1\omega_1'$. Moreover, the action of $\bar{\mathfrak{h}}$ on U contains the scalars. This implies that we may replace $(\mathfrak{sl}(m+n)+\tilde{\mathfrak{g}}, \mathfrak{sl}(m)+\mathfrak{sl}(n)+\tilde{\mathfrak{h}}, V)$ by $(\mathfrak{sl}(m)+\mathfrak{sl}(n)+\tilde{\mathfrak{g}}, \mathfrak{sl}(m)+\mathfrak{sl}(n)+\tilde{\mathfrak{h}}, \omega_1\omega_1' \oplus V)$. Graphically:



Taking also the degenerate case $n = 1$ into account we see from Table 1 that only the triples $(\mathfrak{sl}(m+1)+\mathfrak{sl}(k), \mathfrak{sl}(m)+\mathfrak{sl}(k), \omega_1\omega_1')$, $k \geq 1$, $(\mathfrak{sl}(m+1), \mathfrak{sl}(m), \omega_2)$, and $(\mathfrak{sl}(m+n), \mathfrak{sl}(m)+\mathfrak{sl}(n), \omega_1)$ are spherical (for $m \geq 3$).

8.2. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sl}(2n), \mathfrak{sp}(2n))$, $n \geq 2$**:** we have $\mathfrak{l} = \mathfrak{sl}(2)^n$ and we can glue the following modules:

8.2.1. $(\mathfrak{sp}(2n), \omega_1)$*:* the glued triple is indeed spherical.

8.2.2. $(\mathfrak{sp}(4), \omega_2)$ *with* $n = 4$*:* the dimension of V is too big.

8.2.3. $(\mathfrak{sp}(2n)+\mathfrak{sl}(m), \omega_1\omega_1')$ *with* $n \geq 3$*,* $m \geq 2$*:* the module $(\mathfrak{l}, V)$ is not in Table 1.

8.2.4. $(\mathfrak{sp}(4)+\mathfrak{sl}(m), \omega_1\omega_1')$ *with* $n = 4$*,* $m \geq 2$*:* if $m \geq 3$, $(\mathfrak{l}, V)$ is not in Table 1. For $m = 2$, the dimension of V is too big.

This exhausts all irreducible V. The only case left to check is

8.2.5. $(\mathfrak{sp}(2n), \omega_1+\omega_1)$*:* the dimension of V is too big.

8.3. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sp}(2m+2n), \underline{\mathfrak{sp}(2m)} + \mathfrak{sp}(2n))$, $m \geq 2, n \geq 1$**:** if $m \leq n+1$ then the image of $\mathfrak{l}$ in $\mathfrak{sp}(2m)$ is $\mathfrak{sl}(2)^m$ and we are back to case 8.2 with only $(\mathfrak{sp}(2m+2n), \mathfrak{sp}(2m)+\mathfrak{sp}(2n), \omega_1)$ being spherical. Therefore assume $m > n+1$. Since, in particular, $m \geq 3$ we have to check the following cases:

8.3.1. $(\mathfrak{sp}(2m), \omega_1)$*:* spherical.

8.3.2. $(\mathfrak{sp}(2m)+\mathfrak{sl}(k),\omega_1\omega_1')$ *with* $k=2,3$*:* here $(\mathfrak{l},V)$ is not in Table 1 unless $n=1$ and $k=2$. Also in that case, V is not a spherical $\mathfrak{l}+\mathfrak{z}$-module (see Lemma 5.4).

After this, the only reducible V to check is

8.3.3. $(\mathfrak{sp}(2m),\omega_1+\omega_1)$*:* not spherical (see Lemma 5.4).

8.4. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathfrak{so}(n+1),\mathfrak{so}(n))$**,** $n\geq 7$**:** in this case $\mathfrak{l}=\mathfrak{so}(n-1)$ and the following possible gluings arise:

8.4.1. $(\mathfrak{so}(n),\omega_1)$*:* $V|_{\mathfrak{l}+\mathfrak{z}}=\mathbb{C}^{n-1}\oplus\mathbb{C}$ is not spherical.

8.4.2. $(\mathfrak{so}(n),\text{spin rep.})$*:* according to Table 1 we must have $n=7$, 8, 9, or 10. All of them yield spherical triples except for $n=9$ (see Lemma 5.4). As

for reducible representations we have:

8.4.3. $(\mathfrak{so}(8),\omega_3+\omega_4)$*:* not spherical since $V|_{\mathfrak{l}}$ has multiplicities.

8.5. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathfrak{so}(n+2),\mathfrak{so}(n))$**,** $n\geq 5$**:** We can use Lemma 5.8 with $U=\omega_1$. This implies $n=8$ and $V=\text{spin rep.}$

8.6. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathfrak{so}(2n),\mathfrak{sl}(n))$**,** $n\geq 4$**:** using Lemma 5.8 with $U=\omega_2$ we see that only $(\mathfrak{so}(2n),\mathfrak{sl}(n),\omega_1)$ is spherical.

8.7. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathsf{G}_2,\mathfrak{sl}(3))$**:** in this case we can apply Lemma 5.8 with $U=\omega_1$. Since $\mathfrak{s}=0$ only the implication (1) $\Rightarrow$ (2) is valid. This leaves the following cases to check:

8.7.1. $(\mathfrak{sl}(3),\omega_1)$*:* spherical.

8.7.2. $(\mathfrak{sl}(3)+\mathfrak{sl}(n),\omega_1\omega_1')$*,* $n\geq 2$*:* not spherical (see Lemma 5.4).

8.8. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathfrak{sl}(n)+\mathfrak{sl}(n),\mathfrak{sl}(n))$**,** $n\geq 3$**:** when only $\mathfrak{sl}(n)$ acts on V then $\mathfrak{l}=\mathfrak{t}^{n-1}$, hence $\dim V\leq n$. This implies $(\mathfrak{h},V)=(\mathfrak{sl}(n),\omega_1)$ which indeed is spherical for $\mathfrak{t}^{n-1}+\mathfrak{t}^1$. Otherwise, V has an irreducible component of the form $(\mathfrak{sl}(n)+\mathfrak{k},\omega_1\otimes U)$ with $\dim U\geq 2$ such that $\mathbb{C}^n\otimes U$ is spherical for $\mathfrak{t}^n+\mathfrak{k}$. This does not exist for $n\geq 3$ as Table 1 shows there are no indecomposable spherical modules with more than two irreducible (fiber) components.

8.9. $(\mathfrak{g}_0,\mathfrak{h}_0)=(\mathfrak{sp}(2m+2)+\mathfrak{sp}(2n+2),\underline{\mathfrak{sp}(2m)}+\mathfrak{sl}(2)+\mathfrak{sp}(2n))$**,** $m\geq 2$**,** $n\geq 0$**:** after cutting the diagram at the $\mathfrak{sl}(2)$-vertex of $\mathfrak{h}$ (Lemma 5.11) we see from case 8.3 that at most the triple $(\mathfrak{sp}(2m+2)+\mathfrak{sp}(2n+2),\mathfrak{sp}(2m)+\mathfrak{sl}(2)+\mathfrak{sp}(2n),\omega_1)$ is spherical, and it is.

9. Tree-like extensions

In this section, we classify all reduced spherical triples whose diagram is a tree, i.e., contains no cycles. These triples are called *tree-like.*

Lemma 9.1. *Let $(\mathfrak{g}, \mathfrak{h}, V)$ be a reduced tree-like spherical triple with exactly one base component. Then it appears in Table 2.*

Proof. Let $(\mathfrak{g}_0, \mathfrak{h}_0)$ be the base component. We handle the case of exactly two fiber components first. Since they have to be glued to different simple factors of $\mathfrak{h}_0$ we see that $\mathfrak{h}_0$ cannot be simple. Moreover, $(\mathfrak{g}_0, \mathfrak{h}_0)$ must have at least one simple extension. That leaves the following possibilities:

9.1. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sl}(m+n), \mathfrak{sl}(m) + \mathfrak{sl}(n))$**,** $m, n \geq 2$**:** Again, we can use Lemma 5.8. Then we are done, since there are no indecomposable spherical modules with more than two irreducible components.

9.2. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sp}(2m+2n), \mathfrak{sp}(2m) + \mathfrak{sp}(2n))$**,** $m, n \geq 1$**:** Let $V = V' \oplus V''$ where V' and V'' are attached to $\mathfrak{sp}(2m)$ and $\mathfrak{sp}(2n)$, respectively.

9.2.1. *Case $m \geq n \geq 2$:* then $V' = \omega_1$ and $V'' = \omega_1'$. Hence, V contains as an $\mathfrak{l}$-submodule $(\mathfrak{sl}(2)^n, (\mathbb{C}^2 + \mathbb{C}^2)^n)$ which is not spherical for dimension reasons.

9.2.2. *Case $m > n = 1$:* then $V' = \omega_1 = \mathbb{C}^{2m-2} \oplus \mathbb{C}^2$. The $\mathfrak{sl}(2)$-factor in $\mathfrak{l}$ acts diagonally on $\mathbb{C}^2 \subseteq V'$ and on V''. Thus, we can attach a representation V'' if and only if $\mathbb{C}^2 \oplus V''$ is a spherical $\mathfrak{l} + \mathfrak{z}$-module. These are precisely the representations $(\mathfrak{sl}(2) + \mathfrak{sl}(n), \omega_1\omega_1')$, $n \geq 1$ and $(\mathfrak{sl}(2) + \mathfrak{sp}(2n), \omega_1\omega_1')$, $n \geq 2$ (as in the proof of Lemma 8.3).

9.2.3. *Case $m = n = 1$:* in this case $\mathfrak{l} = \mathfrak{sl}(2)$ acting diagonally on V' and V''. Therefore, V' and V'' are two indecomposable modules which, when branched to $\mathfrak{l}$, are "glued" at one $\mathfrak{sl}(2)$-vertex yielding the last item of Table 2.

9.3. $(\mathfrak{g}_0, \mathfrak{h}_0) = (\mathfrak{sp}(2m+2) + \mathfrak{sp}(2n+2), \mathfrak{sp}(2m) + \mathfrak{sl}(2) + \mathfrak{sp}(2n))$**,** $m, n \geq 0$**:** the principal subalgebra of the pair contains a factor $\mathfrak{t}^1$. Looking at all simple extensions one sees that each irreducible component of V contains an $\mathfrak{l}$-submodule of the form $\mathbb{C}^2 \otimes \mathbb{C}^k$, $k \geq 1$, where $\mathfrak{t}^1 \subseteq \mathfrak{l}$ acts only via its embedding into $\mathfrak{sl}(2)$ and where $\mathbb{C}^k$ is acted on by either $\mathfrak{sl}(k)$ or $\mathfrak{sp}(k)$. Since there are two fiber components we have another submodule $\mathbb{C}^2 \otimes \mathbb{C}^l$, $l \geq 1$. The $\mathfrak{t}^1$-factor acts diagonally on the two submodules of V. Since $(\mathbb{C}^2 \otimes \mathbb{C}^k) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^l)$ is not a spherical module for $\mathfrak{t}^1 + \mathfrak{sl}(k) + \mathfrak{sl}(l) + \mathfrak{z}$ we conclude that $(\mathfrak{g}_0, \mathfrak{h}_0)$ has no extensions with two fiber components.

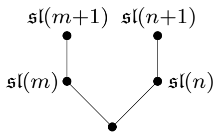
Finally, none of the triples with two fiber components has a "free" factor in $\mathfrak{h}_0$. Therefore, there are no reduced triples with more than two fiber components. □

Lemma 9.2. *There are no reduced tree-like spherical triples with two base components.*

Proof. Let $(\mathfrak{g}, \mathfrak{h}, V)$ be a counterexample whose graph Γ has a minimal number of edges. Since Γ is connected there must be a fiber-component F which is connected to two base components B_1 and B_2. One may erase the edges leading to other components (Corollary 5.12). Hence minimality implies that Γ has only the three components B_1, B_2 and F. Since Γ is a tree the intersection $F \cap B_i$ consists of a single vertex $\mathfrak{h}_i$.

Let Γ_i be the union of F with B_i. Then cutting Γ at $\mathfrak{h}_{3-i}$ (Lemma 5.11) shows that Γ_i is a connected component of a spherical triple hence spherical as well.

Lemma 9.1 says that we can find Γ_i in Table 2. Since exactly two simple factors, $\mathfrak{h}_1$ and $\mathfrak{h}_2$, act on V we see that V is either $(\mathfrak{sl}(2) + \mathfrak{sp}(2n), \omega_1\omega_1')$ with $n \geq 2$ or $(\mathfrak{sl}(m) + \mathfrak{sl}(n), \omega_1\omega_1')$ with $m, n \geq 2$. We can rule out the first case since the $\mathfrak{sp}(2n)$-factor cannot be attached to anything. When $m, n \geq 3$, the second case leads to a unique triple namely



For $m, n \geq 2$ none of these triples is spherical. If $m = 2$ or $n = 2$ other base components (namely those from Lemma 8.3) could be attached. All of them just act through a factor $\mathfrak{t}^1$ in $\mathfrak{l}$. Therefore, these triples are not spherical either. □

Combining Lemmas 9.1 and 9.2 we get:

Corollary 9.3. *Let* $(\mathfrak{g}, \mathfrak{h}, V)$ *be a reduced tree-like spherical triple with* $\mathfrak{g} \neq \mathfrak{h}$ *and* $V \neq 0$. *Then it appears in Table 2.*

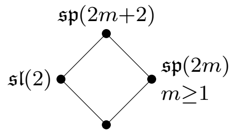
10. Cycles

The following lemma finishes off the classification:

Lemma 10.1. *Every spherical triple is tree-like.*

Proof. Let $(\mathfrak{g}, \mathfrak{h}, V)$ be a counterexample whose diagram Γ has minimal number of edges. Clearly, the triple is reduced. Let C be a cycle. Since, by inspection, all base components $(\mathfrak{g}_0, \mathfrak{h}_0)$ are trees, C has to contain a vertex V_k. Moreover, again by inspection, at most, hence exactly two factors of $\mathfrak{h}$ act non-trivially on V_k. Thus $V_k = U_1 \otimes U_2$. Let $\tilde{\Gamma}$ be the triple obtained by cutting Γ at V_k (Lemma 5.10), i.e., where we replace $U_1 \otimes U_2$ by $U_1 \oplus U_2$. Then $\tilde{\Gamma}$ is a tree since otherwise we could erase the edges adjacent to U_i and obtain a smaller spherical triple whose diagram contains a cycle. This implies that $\tilde{\Gamma}$ is a member of Table 2. Moreover, it contains two vertices in the V-layer which are adjacent to exactly one vertex each in the $\mathfrak{h}$-layer.

There is only one such case which results in



for Γ. As a module for $\tilde{\mathfrak{l}} = \mathfrak{sp}(2)+\mathfrak{sp}(2m-2)+\mathfrak{t}^1$, V contains $(\mathfrak{gl}(2), \mathbb{C}^2\otimes\mathbb{C}^2)$ which is not spherical, by dimension. $\square$

TABLE 1. Primitive spherical modules

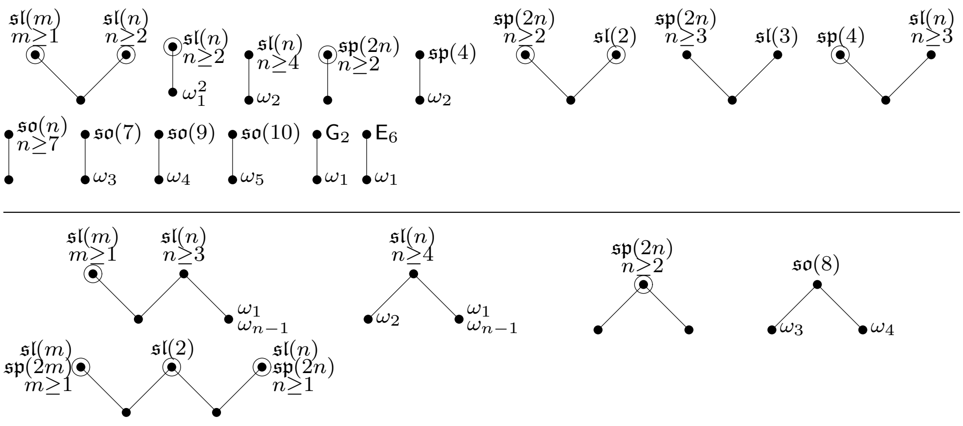


TABLE 2. Reduced spherical triples $(\mathfrak{g}, \mathfrak{h}, V)$, $\mathfrak{g} \neq \mathfrak{h}$, $V \neq 0$

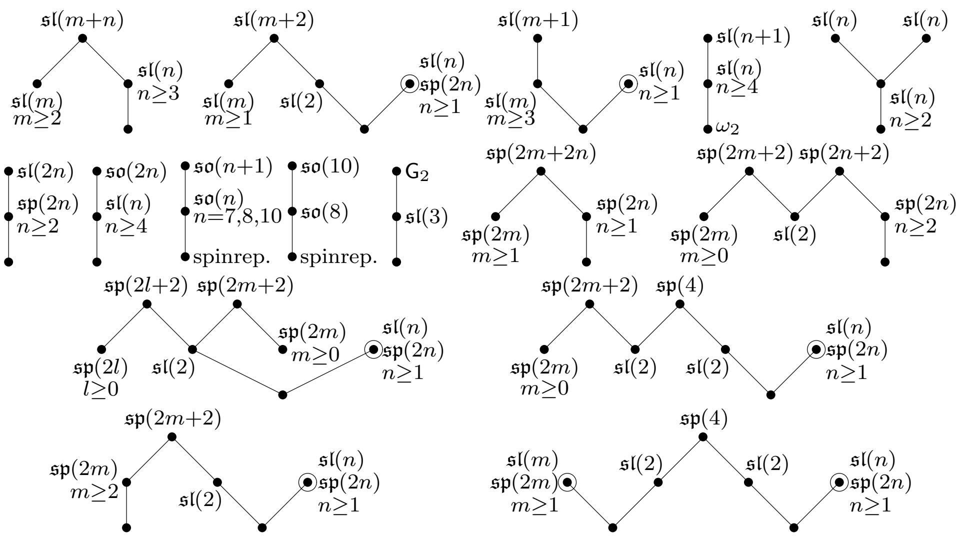


TABLE 3. Inference Rules

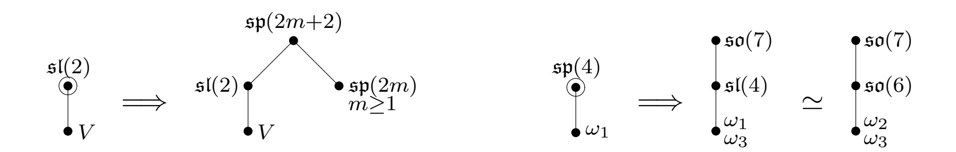


Table 4. Table of all primitive spherical pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple. A "$*$" means that $\mathfrak{s}$ is non-trivial in which case $\mathfrak{s} = \mathfrak{t}^1$. The third column lists the principal subalgebra $\mathfrak{l}$ of the pair and in some cases an indication of its embedding into $\mathfrak{h}$. The last column contains marks for reference in the proof of Proposition 6.1.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{l}$		
$\mathfrak{sl}(m+n)$	$*\mathfrak{sl}(m) \times \mathfrak{sl}(n)$	$\mathfrak{gl}(m-n) \times \mathfrak{t}^{n-1}$	$m \geq n \geq 1$	$\bullet$
$\mathfrak{sl}(2n)$	$\mathfrak{sp}(2n)$	$\mathfrak{sl}(2)^n$	$n \geq 2$	$\bullet$
$\mathfrak{sl}(2n+1)$	$*\mathfrak{sp}(2n)$	$\mathfrak{t}^1$	$n \geq 2$	b
$\mathfrak{sl}(n)$	$\mathfrak{so}(n)$	0	$n \geq 3$	b
$\mathfrak{sp}(2m+2n)$	$\mathfrak{sp}(2m) \times \mathfrak{sp}(2n)$	$\mathfrak{sp}(2m-2n) \times \mathfrak{sl}(2)^n$	$m \geq n \geq 1$	$\bullet$
$\mathfrak{sp}(2n+2)$	$*\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n-2)$	$n \geq 1$	$+$
$\mathfrak{sp}(2n)$	$*\mathfrak{sl}(n)$	0	$n \geq 2$	b
$\mathfrak{so}(m+n)$	$\mathfrak{so}(m) \times \mathfrak{so}(n)$†	$\mathfrak{so}(m-n)$	$\begin{cases} m \geq n \geq 1 \\ m+n \geq 7 \end{cases}$	$\circ$
$\mathfrak{so}(2n)$	$*\mathfrak{sl}(n)$	$\begin{cases} \mathfrak{sl}(2)^m & ; n = 2m \\ \mathfrak{sl}(2)^m \times \mathfrak{t}^1 ; & n = 2m+1 \end{cases}$	$n \geq 4$	$\bullet$
$\mathfrak{so}(2n+1)$	$*\mathfrak{sl}(n)$	0	$n \geq 3$	b
$\mathfrak{so}(7)$	G_2	$\mathfrak{sl}(3)$		b
$\mathfrak{so}(8)$	G_2	$\mathfrak{sl}(2)$		b
$\mathfrak{so}(9)$	$\mathfrak{so}(7)$‡	$\mathfrak{sl}(3)$		b
$\mathfrak{so}(10)$	$*\mathfrak{so}(7)$‡	$\mathfrak{sl}(2)$		b
G_2	A_2	A_1		$\bullet$
G_2	$\mathsf{A}_1 \times \mathsf{A}_1$	0		b
F_4	B_4	B_3		x
F_4	$\mathsf{C}_3 \times \mathsf{A}_1$	0		b
E_6	C_4	0		b
E_6	F_4	D_4		x
E_6	$*\mathsf{D}_5$	$\mathsf{A}_3 \times \mathfrak{t}^1$		x
E_6	$\mathsf{A}_5 \times \mathsf{A}_1$	$\mathfrak{t}^2 \subseteq \mathsf{A}_5$		b'
E_7	$*\mathsf{E}_6$	D_4		b
E_7	A_7	0		b
E_7	$\mathsf{D}_6 \times \mathsf{A}_1$	$\mathsf{A}_1^3 \subseteq \mathsf{D}_6$		b'
E_8	D_8	0		b
E_8	$\mathsf{E}_7 \times \mathsf{A}_1$	$\mathsf{D}_4 \subseteq \mathsf{E}_7$		b'

†Read $*\mathfrak{so}(m)$ if $n = 2$.

‡Embedding via $\mathfrak{so}(7) = \mathfrak{spin}(7) \hookrightarrow \mathfrak{so}(8) \hookrightarrow \mathfrak{so}(8+\varepsilon)$.

TABLE 5. Table of all primitive spherical pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ not simple. The "$*$" indicates $\mathfrak{s} \neq 0$ and its embedding into $\mathfrak{g}$. The second column lists the principal subalgebra $\mathfrak{l}$ of the pair. Again, the last column contains marks for reference in the proof of Lemma 6.1.

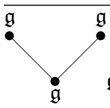
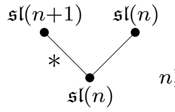
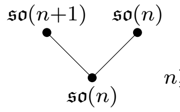
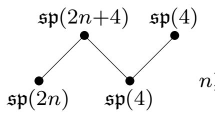
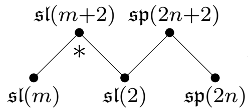
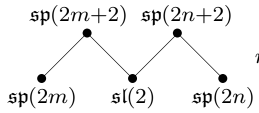
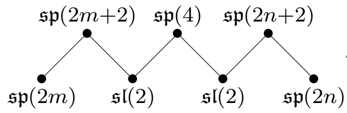
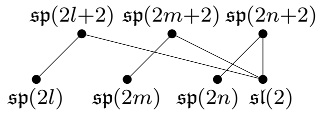
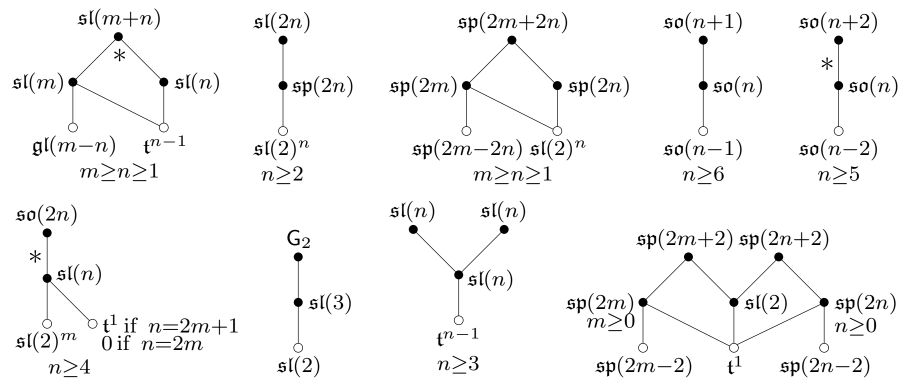
$(\mathfrak{g}, \mathfrak{h})$	$\mathfrak{l}$	
 $\mathfrak{g}$ simple	max. torus	$\circ$
 $n \geq 2$	0	b
 $n \geq 5$	0	b
 $n \geq 1$	$\mathfrak{sp}(2n-4)$	b'
 $m \geq 1$, $n \geq 0$	$\mathfrak{gl}(m-2) \times \mathfrak{sp}(2n-2)$	x
 $m \geq 0$, $n \geq 0$	$\mathfrak{sp}(2m-2) \times \mathfrak{t}^1 \times \mathfrak{sp}(2n-2)$	$\bullet$
 $m \geq 0$, $n \geq 0$	$\mathfrak{sp}(2m-2) \times \mathfrak{sp}(2n-2)$	b'
 $l \geq 0$, $m \geq 0$, $n \geq 0$	$\mathfrak{sp}(2l-2) \times \mathfrak{sp}(2m-2) \times \mathfrak{sp}(2n-2)$	b'

TABLE 6. Table of base components of primitive spherical triples with $V \neq 0$ and their principal subalgebras. A "$*$" indicates $\mathfrak{s} \neq 0$.



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Friedrich Knop, Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway NJ 08854-8019, USA
E-mail address: `knop@math.rutgers.edu`

Bart Van Steirteghem, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
E-mail address: `bvans@math.ist.utl.pt`