1. (a) Let $\sigma : V \rightarrow \{R, B\}$ be chosen at random with, for each $v \in V$, $\Pr(\sigma(v) = R) = p$, independent of other choices ($p$ TBA). Let

$$Q_i = \{\sigma(x) = R \forall x \in A_i, \sigma(y) = B \forall y \in B_i\}.$$  

Then our hypotheses imply $Q_i \cap Q_j = \emptyset \forall i \neq j$; so

$$1 \geq \sum \Pr(Q_i) = m p a (1 - p)^b,$$

and taking $p = a/(a + b)$ (the value that minimizes $p^{-a}(1 - p)^{-b}$) gives the desired bound.

(b) (Due to Ahlswede and Zhang.)

Claim. For any $\pi = \pi(1), \ldots, \pi(n) \in S_n$, there’s a unique $i \in [n]$ satisfying

(i) $X := \{\pi(1), \ldots, \pi(i)\} \in \langle A\rangle$,
(ii) $\pi(i) \in Z(X)$.

This implies that $n!$ is the number of ways to choose $X \in \langle A\rangle$ and then a permutation $\pi$ with

(iii) $\{\pi(1), \ldots, \pi(|X|)\} = X$ and
(iv) $\pi(|X|) \in Z(X)$.

But for any $X \in \langle A\rangle$, the number of ways to choose $\pi$ satisfying (iii) and (iv) is $|Z(X)|(|X| - 1)!(n - |X|)!$ (so the claim suffices).

Proof of claim. Uniqueness: if $X = \{\pi(1), \ldots, \pi(i)\} \in \langle A\rangle$ and $Y = \{\pi(1), \ldots, \pi(j)\}$ with $j > i$, then $Z(Y) \subseteq Z(X)$ $(\subseteq X)$ implies $\pi(j) \notin Z(Y)$ (i.e. $Y$ fails (iv)).

Existence: Let $i$ be minimum with $X := \{\pi(1), \ldots, \pi(i)\} \in \langle A\rangle$. Then since $\{\pi(1), \ldots, \pi(i - 1)\} \notin \langle A\rangle$, we have $X \supseteq A \in A \Rightarrow \pi(i) \in A$, i.e. $\pi(i) \in Z(X)$.

2. If the elements of $P$ are $x_1, \ldots, x_n$, let $G$ be the bigraph on $\{v_1, \ldots, v_n\} \cup \{w_1, \ldots, w_n\}$ with $v_i \sim w_j$ iff $x_i < x_j$. We just need

Claim. (i) $\beta(P) \leq n - \nu(G)$, and (ii) $w(P) \geq n - \tau(G)$.

(Then König’s Theorem gives $\beta(P) \leq w(P)$, which is what we want.)

Proof of (i). With a matching $M$ of $G$ associate the natural chain partition of $P$: $x_i \prec x_j$ in some chain if $v_iw_j \in M$. Then $x_i$ is maximal in its chain
iff $M$ does not cover $v_1$, so the number of chains is $n - |M|$ (and then take $M$ of size $\nu(G)$).

Proof of (ii). If $\{v_i : i \in I\} \cup \{w_j : j \in J\}$ is a (vertex) cover of $G$, then $\{x_k : k \in [n] \setminus (I \cup J)\}$ is an antichain of $P$ of size at least $n - |I \cup J|$; in particular, a cover of size $\tau(G)$ gives an antichain of size at least $n - \tau(G)$.

3. (a) Form a new bigraph $G^* = (X^* \cup Y^*, E^*)$ with

$$X^* = \{(x, i) : x \in X, i \in [\alpha(x)]\}, \quad Y^* = \{(y, j) : y \in Y, j \in [\beta(y)]\}$$

and $\{(x, i), (y, j)\} \in E^*$ iff $\{x, y\} \in E$, and check:

(i) condition (1) in the problem implies (actually is equivalent to) Hall’s condition for $G^*$, and

(ii) an $X^*$-perfect matching of $G^*$ (existence of which is implied by (i) and Hall’s Theorem) gives the desired $F$ in $G$.

(b) Let $H$ be the natural bigraph on $E \cup V$; namely:

$$e \sim v \text{ in } H \text{ iff } v \text{ is an end of } e \text{ in } G.$$ 

Notice that for any $E' \subseteq E$ and $W \subseteq V$ the set of ends of edges of $E'$, we have $W = N_H(E')$ and

$$|E'| \leq |E(W)| \leq \sum \{k(v) : v \in W\}.$$ 

Thus we have condition (1) of (a) with $\alpha \equiv 1$ and $\beta = k$, whence (according to (a)) there is some $F \subseteq E(H)$ with $d_F(e) = 1 \forall e \in E$ and $d_F(v) \leq k(v) \forall v \in V$. This gives (actually, is equivalent to) the desired orientation: just orient $e$ away from $v$ if $F$ joins $e$ to $v$.

4. It’s enough to show that there is a $\varphi : \Gamma \to \{X, Y\}$ so that

$$S(v) \cap \varphi^{-1}(X) \neq \emptyset \quad \forall v \in X \quad \text{and} \quad S(v) \cap \varphi^{-1}(Y) \neq \emptyset \quad \forall v \in y. \quad (1)$$

(We can then assign each $v \in X$ a color from $S(v) \cap \varphi^{-1}(X)$ and similarly for $v \in Y$.) But for a random (uniform) $\varphi$, our assumption on the $|S(v)|$’s implies that the probability the event in (1) fails is less than $n2^{-\log_2 n} = 1$; so there must be a $\varphi$ for which the event does not fail.

5. Let $X$ be the random leaf reached by the the natural random walk on $T$ that does: (i) start at the root; (ii) at each step for which the current vertex
is not a leaf, move to a uniformly chosen child of this vertex; (iii) stop upon reaching a leaf. If \( v_0, \ldots, v_k = v \) is the path from the root, \( v_0 \), to \( v \), then 
\[
\Pr(X = v) = \prod_{i=0}^{k-1} d(v_i)^{-1} \geq r^{-D(v)}
\]
where \( d(\cdot) \) is number of children and \( D \) is depth, and we have
\[
1 = \sum \Pr(X = v) \geq \sum r^{-D(v)} \geq nr^{-\sum D(v)/n}
\]
(where the sums are over leaves \( v \) and the second inequality follows from convexity of \( r^{-x} \)). The statement in the problem follows.

6. **Observation:** if \( \chi(H) < \chi/t \) then there is a \( W \subseteq V \) with \( |G[W]| \geq |W|t/2 \) and \( W \) independent in \( H \) (where for graphs size means number of edges).

**Proof.** If \( V_1 \cup \cdots \cup V_m \) is a coloring of \( H \) (s. a partition of \( V \) into independent sets) with \( m < \chi/t \), then (clearly) \( \chi(G[V_i]) > t \) for some \( i \); so the proposition in the problem says there is some \( W \subseteq V_i \) with \( \delta(G[W]) \geq t \) and (therefore) \( |G[W]| \geq |W|t/2 \).

But the probability that there is a \( W \) as in the observation (necessarily with \( |W| > t \)) is less than
\[
\sum_{s > t} \left( \begin{array}{c} n \\ s \end{array} \right) 2^{-st/2},
\]
which is \( o(1) \) if \( t = 2 \log n \).

7. It’s ETS that we can couple so that \( X \subseteq Y \); one way:
   (a) choose \( x_1, \ldots, x_{2m} \) as in the problem;
   (b) for \( i \in [m] \) set \( y_i = x_i \) and: if \( x_{m+i} \neq x_i \), set \( z_i = x_{m+i} \); otherwise choose \( z_i \) uniformly (and independently of all other choices) from \( S \setminus \{x_i\} \). Then (check) \( Y \) has the correct distribution (and of course \( X \subseteq Y \)).

8. Let \( X_e = 1_{\{e \in I \}} \) (\( I \) as in the problem), and break the calculation into three parts:
\[
\Var[X] = \sum_e \Var[X_e] + \sum_{|e|f| = 1} \Cov[X_e, X_f] + \sum_{e \cap f = \emptyset} \Cov[X_e, X_f]
\]
(where \( \Cov[Y, Z] = \E[YZ] - \E[Y] \E[Z] \)). Since \( \E[X_e] = o(1) \), \( \Var[X_e] \sim \E[X_e] \) and
\[
\sum_e \Var[X_e] \sim \E[X] \sim \frac{cn}{2e^{2c}}.
\]
For \(|e \cap f| = 1\), we have

\[ \text{Cov}[X_e, X_f] = \mathbb{E}[X_e X_f] - \mathbb{E}[X_e] \mathbb{E}[X_f] = -\mathbb{E}[X_e] \mathbb{E}[X_f] \]

(since \(e, f\) can’t both be isolated). The number of such terms is \(\binom{n}{2} 2(n-2) \sim n^3\), so

\[ \sum_{|e \cap f| = 1} \text{Cov}[X_e, X_f] \sim -n^3(p(1-p)^{2n-4})^2 \sim -n^3\left(\frac{c}{ne^2c}\right)^2 = -n^3e^{-4c}. \]

For \(e \cap f = \emptyset\),

\[ \text{Cov}[X_e, X_f] = \mathbb{E}[X_e X_f] - \mathbb{E}[X_e] \mathbb{E}[X_f] \]

\[ = p^2(1-p)^{4n-12} - (p(1-p)^{2n-4})^2 \]

\[ = p^2(1-p)^{4n-12}(1 - (1-p)^4) \]

(The covariance is nonzero because there are 4 edges that affect both \(X_e\) and \(X_f\).) Now \(1 - (1-p)^4 \sim 4p\) (since \(p = o(1)\)), and the number of \(e, f\) with \(e \cap f = \emptyset\) is \(\binom{n}{2} (n-2)^2 \sim n^4/4\), so

\[ \sum_{e \cap f = \emptyset} \text{Cov}[X_e, X_f] \sim \frac{n^4}{4} \left(\frac{c}{n}\right)^3 e^{-4c} = nc^3e^{-4c}. \]

So, finally,

\[ \text{Var}[X] \sim n \left[ \frac{1}{2} ce^{-2c} - c^2e^{-4c} + c^3e^{-4c} \right]. \]

9. First observe that for \(a, b, x, y\) as in the hint, \((\varphi(x), \varphi(y))\) is uniform from \(W := \mathbb{Z}_p^2 \setminus \{(z, z) : z \in \mathbb{Z}_p\}\). (Equivalently, for \((u, v)\) in \(W\) there’s a unique \((a, b)\) with \(ax + b = u\) and \(ay + b = v\).)

Let \(\xi_x\) be the indicator of \(\{\varphi(x) \in Y\}\), so \(|(aX+b) \cap Y| = \sum_{x \in X} \xi_x = : \xi\). We have \(\mathbb{E}\xi_x = t/p\) (so \(\xi \sim t^2/p\)) and, by the above observation, \(\mathbb{E}\xi_x \xi_y = \frac{t(t-1)}{p^2}\) for distinct \(x, y\), yielding

\[ \sigma^2_\xi = \sum_{x \in X} \mathbb{E}\xi_x + \sum_{x \in X} \sum_{x \neq x' \in X} \mathbb{E}\xi_x \xi_{x'} - t^2/p^2 \]

\[ = \frac{t^2}{p} + \frac{t^2}{p^2} (t-1)^2/(p(p-1)) - t^4/p^2 < t^2/p \]

(the inequality following from our assumption on \(t\); actually the assumption \(t < p - \sqrt{p}\) was unnecessary). Chebyshev’s Inequality thus gives

\[ \Pr(|\xi - \mathbb{E}\xi| \geq t/\sqrt{p}) \leq \sigma^2_\xi/(t^2/p) < 1, \]

so there are \(a, b\) for which \(|\xi - \mathbb{E}\xi| < t/\sqrt{p}\), which is what we want.