1. Verify the assertions in the “binomial coefficients” section of the handout.

2. Slightly modifying the notation from class, define a tournament on $V$ to be $T = (V, A)$ with $A$ (for “arcs”) a subset of $V \times V$ and $xy \in A$ iff $yx \notin A$. A subtournament of $T$ is then $T|_W = (W, A \cap (W \times W))$ for some $W \subseteq V$.

   A tournament $(V, A)$ is transitive if $xy, yz \in A \Rightarrow xz \in A$. Define $v(n)$ to be the largest $k$ such that every tournament on $n$ vertices has a transitive subtournament on $k$ vertices. Show

   (a) $v(n) \geq \lceil \log_2 n \rceil + 1$ (this part not random), and
   (b) $v(n) \leq \lfloor 2\log_2 n \rfloor + 1$.

3. Show that for each $k, l \in \mathbb{N}$ there is a finite graph $G$ on some vertex set $V$ with the property: for any disjoint $K, L \subseteq V$ with $|K| = k, |L| = l$, there is some $v \in V \setminus (K \cup L)$ such that

   $v \sim x \ \forall x \in K \text{ and } v \not\sim y \ \forall y \in L. \quad (1)$

   [This problem is mainly an excuse to mention an old open question, due to G. Cherlin, perhaps among others: does the above statement remain true if we require that $G$ be triangle-free and only require (1) when $K$ is independent? (Easy exercise: this is true if we allow $G$ to be infinite.)]

4. Let $A_1, \ldots, A_n$ be events in a probability space and $\mu = \sum \Pr(A_i)$ (the expected number of $A_i$’s that occur). Let $Q_l$ be the event that some $l$ independent $A_i$’s occur. Show $\Pr(Q_l) \leq \mu^l/l!$.

5. A dominating set in a graph $G$ on $V$ is $U \subseteq V$ such that each vertex of $V \setminus U$ has at least one neighbor in $U$. For any $n$ and $1 < \delta \in \mathbb{N}$, find $\alpha = \alpha(n, \delta)$ as small as you can such that every $G$ on $n$ vertices with minimum degree at least $\delta$ has a dominating set of size at most $\alpha$.

   [Small hint: This is a nice use of linearity of expectation. Bigger hint: It’s also A-S, Theorem 1.2.2. (You could look there for $\alpha$ and then try to show it works.)]

6. Let $H$ be a graph, and suppose there exists an $n$-vertex, $t$-edge graph (say $G$) not containing any copy of $H$. Then if $kt > n^2 \ln n$, it is possible to color $E(K_n)$ using $k$ colors, so that no color contains a copy of $H$.

   [A-S, 2.7.5, but you don’t really need lin. of exp.]
7. Let \( G = (V, E) \) be a graph with \( |V| = n \) and minimum degree \( \delta \). Show there is a partition \( V = A \cup B \) such that each vertex of \( B \) has at least one neighbor in each of \( A, B \), and \( |A| < O\left(\frac{n \ln \delta}{\delta}\right) \).

[A-S, 1.6.4; note they give it a (*).]

Digression:

8. Prove A-S, Theorem 2.2.3 using Inclusion-Exclusion.

[So no randomness in this one. You'll need the version of I-E that gives \( f(\cap A_i) \) when \( f : 2^X \to \mathbb{R}^+ \) is a measure (i.e. \( f(X') = \sum \{ f(x) : x \in X' \} \)).]

The next three aren't much fun, but are good calculation practice.

9. Prove \( R(k, 2k) = \Omega(ka^k) \) with \( \alpha \) as large as you can make it.

10. Use the “deletion method” to improve the constant in Erdős’ lower bound for \( R(k, k) \).

11. Bound \( R(3, t) \) and \( R(4, t) \) from below via the deletion method. Try to optimize the constant factors. [You should get

\[
R(3, t) > (1 - o(1)) \frac{2\sqrt{3}}{3} \left( \frac{t}{\ln t} \right)^{3/2}
\quad \text{and} \quad
R(4, t) > (1 - o(1)) \frac{3\sqrt{6}}{16} \left( \frac{t}{\ln t} \right)^2.
\]
Three plus one on Property B, in increasing order of difficulty:

12. (a) Suppose $A \subseteq [4n], |A| = n$ and we randomly equicolor $[4n] = R \cup B$ ("equicolor" meaning, of course, that $|R| = |B| = 2n$). Find an asymptotic expression for $Pr(A \text{ monochromatic}).$

(b) Use (a) to give a lower bound on the minimum size, say $f(n)$, of an $\mathcal{H} \subseteq \binom{[4n]}{n}$ not having property B. (Your lower bound should be much bigger than our general upper bound for Property B.)

13. As in class (except $n$ was $t$), let $g(n)$ be the least size of an $n$-uniform hypergraph $\mathcal{H}$, on vertex set $V$ say, such that $|V| \geq 2n$ and every set of $|V|/2$ vertices contains some member of $\mathcal{H}$. Show $g(n) > \Omega(n2^n)$. (To avoid irrelevancies, let’s say $|V| = m$ is even.)

[Hint: Apply a deletion method to an appropriate random subset of $V$.]

14. Show that for each $n$ there exists an $n$-uniform hypergraph $\mathcal{H}$ not having property B and satisfying

$$|A \cap B| \leq 1 \quad \forall A, B \in \mathcal{H} \quad (A \neq B).$$

[Hint: Choose edges for $\mathcal{H}$ at random from a suitable vertex set and use a deletion method. You may (or may not) want to use Markov’s Inequality.]

15. One more open (I think) problem related to Property B: what can one say about

$$\min\{|\mathcal{H}| : \mathcal{H} \text{ n-uniform and intersecting, without Property B}\}?$$

(A hypergraph $\mathcal{H}$ is intersecting if $A, B \in \mathcal{H} \Rightarrow A \cap B \neq \emptyset$.) I believe the best known u.b. is $f(n) \leq 7^{(n-1)/2}$ for infinitely many $n$. This is due to Erdős and Lovász (a worthwhile, though nonrandom, exercise, since similar things come up fairly often; try a recursive construction starting with the projective plane of order 2), who ask: does $f(n)^{1/n} \rightarrow 2$?
The next three are instructive, but written proofs can get a little long.

16. For $G = G_{n,1/2}$, show $\chi(G) < (1 + o(1)) \frac{n}{\log_2 n}$ a.s.

[Note: “$f(n) < (1 + o(n)) g(n)$ a.s.” means $f(n) < (1 + \varepsilon(n)) g(n)$ a.s. for some $\varepsilon(n) = o(1)$.

Exercise: this is equivalent to “$\forall \varepsilon > 0, f(n) < (1 + \varepsilon(n)) g(n)$ a.s.” (This exercise also justifies the passage to a fixed $\varepsilon$ in the next problem.)]

[Hint: try a greedy coloring.]

17. Here’s a way of showing that the Hajós number, $\sigma(G)$, of $G = G_{n,1/2}$ (the largest $m$ for which $G$ contains a topological $K_m$) is a.s. at least $(2 - o(1)) \sqrt{n}$ (which asymptotically matches the upper bound proved in class).

Fix $\varepsilon > 0$ and set $m = m(n) = (2 - \varepsilon) \sqrt{n}$. Given $W \in \binom{V}{m}$, define a random bipartite graph $H$ on $X \cup (V \setminus W)$, with $X = \{\{x, y\} \subseteq W : x \not\sim_G y\}$ (so $X$ is random), and

$$\{x, y\} \sim_H v \iff x, y \sim_G v.$$  

(Thus $v \in V \setminus W$ is adjacent (in $H$) to $\{x, y\} \in X$ if $(x, v, y)$ is a path that could substitute for the missing edge $xy$.)

Then $G$ contains a topological $K_m$ with node set $W$ provided

(*) $H$ contains an $X$-perfect matching

(i.e. a matching that uses all vertices of $X$). Show $H$ a.s. satisfies ($*$).

[You’ll want to take a look at Hall’s Theorem on matchings in bipartite graphs, if you don’t already know it.]

18. (a) With $k, s$ integers, let $X = \{x_1, \ldots, x_k\}$ and $Y$ be disjoint sets with $|Y| = m = ks$. Let $G$ be the random bipartite graph on $X \cup Y$ in which $\Pr(xy \in G) = 1/2 \forall x \in X, y \in Y$, these choices made independently. Show that if $k = \omega(\log m)$, then there is a.s. a partition $Y = Y_1 \cup \cdots \cup Y_k$ with $|Y_i| = s$ $\forall i$ and

for each $i$, $x_i \sim y \forall y \in Y_i$.

[Hint: Form $H$ from $G$ by replacing each $x_i$ by $x_{i1}, \ldots, x_{is}$ with $x_{ij} \sim_H y$ iff $x_i \sim_G y$. Then $G$ has a partition as above iff $H$ has a perfect matching. So the problem is really to show that $H$ a.s. satisfies Hall’s condition; one way: consider neighborhoods of vertices and pairs of vertices from $X$ and vertices from $Y.$]
(b) Recall that $H$ is a \textit{minor} of $G$ (written $H \preceq G$) if it can be obtained from $G$ by deleting and contracting edges and deleting vertices; equivalently—and more usefully here—if there are disjoint \textit{connected} subgraphs $G_v$ of $G$, $v \in V(H)$, with

$$v \sim_H w \implies E(G_v, G_w) \neq \emptyset.$$ 

Define the \textit{Hadwiger number} of $G$ to be $\tau(G) = \max \{m : G \succ K_m\}$.

For $G = G_{n,1/2}$, show that there are positive constants $c_1$, $c_2$ such that

$$c_1 \frac{n}{\sqrt{\log n}} < \tau(G) < c_2 \frac{n}{\sqrt{\log n}} \quad \text{a.s.}$$

(In particular Hadwiger’s Conjecture is true for a.a. graphs.)

[The lower bound is the harder part; try doing it with each $G_v$ consisting of an $x_v \in V(G)$ and some of its neighbors.]
In contrast to the last few problems, the things on this page are all pretty easy once found. The first two depend on the “Law of Total Probability,” so call for astute choices of conditioning. (The second is a particularly striking example, if you haven’t seen it.)

19. Suppose \( A_1, \ldots, A_m, B_1, \ldots, B_m \) are events with the \( A_i \)'s independent of the \( B_i \)'s and let \( X = \bigcup_{i=1}^{m} (A_i B_i) \), \( Y = \bigcup_{i=1}^{m} A_i \).

Show \( \Pr(X) \geq \Pr(Y) \min_i \Pr(B_i) \).

20. (a) Let \( \mathcal{H} \) be an arbitrary hypergraph on \( S, |S| = n \), and let \( w \) be chosen uniformly at random from \([k]^S\) (functions from \( S \) to \([k]\)). Show, with \( w(A) = \sum_{x \in A} w(x) \), that

\[
\Pr(\exists A \in \mathcal{H}, w(A) < w(B) \forall B \in \mathcal{H} \setminus \{A\}) \geq 1 - n/k.
\]

[In words, the event in question is: there’s a unique \( A \) that attains \( \min \{w(A) : A \in \mathcal{H} \} \). Of course the statement is only interesting when \( k \) is somewhat larger than \( n \) (e.g. \( k = 2n \)).]

(b) Improve to \( 1 - n/(2k) \).

21. Let \( S \) be a finite set and \( m \in \mathbb{P} \), and consider two experiments:

(a) \( x_1, \ldots, x_{2m} \) are chosen uniformly and independently from \( S \);

(b) \( \{y_1, z_1\}, \ldots, \{y_m, z_m\} \) are chosen uniformly and independently from \( \binom{S}{2} \).

Set \( X = \{x_1, \ldots, x_{2m}\} \) and \( Y = \{y_1, \ldots, y_m, z_1, \ldots, z_m\} \), and prove the rather obvious fact that \( \Pr(|Y| \geq t) \geq \Pr(|X| \geq t) \ \forall t \).

22. For events \( A_1, \ldots, A_n \) in a probability space, with \( \mu = \sum \Pr(A_i) \),

\[
\Pr(\text{some } \mu + t \text{ independent } A_i \text{'s occur}) \leq \exp[-\mu \varphi(t/\mu)] \\
\leq \exp[-t^2/(2(\mu + t/3))],
\]

where \( \varphi(x) = (1 + x) \log(1 + x) - x \) for \( x \geq -1 \) (so \( \varphi(-1) = 1 \)).

[Note this improves Problem 4. The assignment here is really the first inequality; the second bound (a little calculus exercise, or see p. 27 of [JLR]) is there to make sense of the first. Hint: consider, for suitable \( k \), the number of sequences of \( k \) independent events that occur and use Markov. You’ll eventually want to bound a sum by an integral.]

\(^{1}\)Janson, Łuczak, Ruciński, Random Graphs
Another that’s easy once you find it:

23. Show that (as mentioned in class) for any sequence of sets \( \{X_n\} \) and increasing \( F_n \subseteq 2^{X_n} \), \( p_c(F_n) \) is a threshold in the Erdős-Rényi sense. You can use the following sequence-avoiding formulation:

For each \( \varepsilon > 0 \) there is an \( m \) such that for any finite set \( X \) and increasing \( F \subseteq 2^X \), with \( p = p_c(F) \): if \( q > mp \) then \( \mu_q(F) > 1 - \varepsilon \) and if \( q < p/m \) then \( \mu_q(F) < \varepsilon \).

24. Let \( p = c/n \) (\( c \) a positive constant), and let \( X \) be the number of isolated edges in \( G = G_{n,p} \). (An edge is isolated if it meets no other edges.) Give asymptotics for the mean and variance of \( X \).

[Note: this is not hard, but people tend to get it wrong: be careful not to throw away terms too soon.]

25. Suppose the r.v. \( X = X^{(n)} \) is \( \mathbb{N} \)-valued and the sequence \( \{p_k := \mathbb{P}(X = k)\}_{k \geq 0} \) is log-concave with no internal zeros (that is, \( p_k^2 \geq p_{k-1}p_{k+1} \forall k \) and \( \{k : p_k \neq 0\} \) is an interval). Then \( \mathbb{E}X \rightarrow \infty \) (as \( n \rightarrow \infty \)) implies \( \mathbb{P}(X = 0) \rightarrow 0 \).

[Hint: WMA (why?) \( p_0 \neq 0 \).

Let \( \alpha = p_0 \) and compare with the (shifted) geometric distribution \( p_k^* = \alpha(1 - \alpha)^k \).

You may find some use for the (useful) identity \( \mathbb{E}X = \sum_k \mathbb{P}(X \geq k) \) (valid for \( \mathbb{N} \)-valued \( X \)).]

26. Let \( M = M(n) \) be the least (positive) integer such that there are \( a_1, \ldots, a_n \in [M] \) for which the \( 2^n \) subset sums \( \sum_{i \in I} a_i, I \subseteq [n] \), are distinct. Use the second moment method to show that \( M(n) = \Omega(2^n/\sqrt{n}) \).

[It’s easy to see \( M(n) \leq 2^{n-1} \). An old ($500) conjecture of Erdős says that \( M(n) = \Omega(2^n) \).]

27. For \( i \in [n] \), let \( v_i = (x_i, y_i) \in \mathbb{Z}^2 \) with each of \( |x_i|, |y_i| \) at most \( 2^{n/2}/(100\sqrt{n}) \). Show that there are disjoint \( I, J \subseteq [n] \) with \( \sum_{i \in I} v_i = \sum_{i \in J} v_i \).

[A-S, 4.8.5]

The next two are somewhat tougher (a.k.a. more interesting).

28. Show that there is a positive constant \( c \) for which the following holds. If \( a_1, \ldots, a_n \in \mathbb{R} \) satisfy \( \sum a_i^2 = 1 \), and \( \varepsilon_1, \ldots, \varepsilon_n \) are chosen uniformly and independently from \( \{\pm 1\} \), then \( \Pr(|\sum \varepsilon_i a_i| \leq 1) \geq c. \)
[A-S, 4.8.2 (with a (†)). It’s conjectured that this is true with \( c = 1/2 \)—which would be best possible, right?]

29. Show that there is a positive constant \( c \) for which the following holds. If \( a_1, \ldots, a_n \in \mathbb{R}^2 \) satisfy \( \sum \|a_i\|^2 = 1 \) and \( \|a_i\| \leq 1/10 \) \( \forall i \) (where \( \| \cdot \| \) is Euclidean norm), and \( \varepsilon_1, \ldots, \varepsilon_n \) are chosen uniformly and independently from \( \{\pm 1\} \), then \( \Pr(\| \sum \varepsilon_i a_i \| \leq 1/3) \geq c \).

[A-S, 4.8.3, with another (†)]

30. Suppose \( A_i = A_i^{(n)} \) are independent events with indicators \( X_i \) and \( \Pr(A_i) = p_i \), and set \( X = \sum X_i \). Show for positive real \( \mu \) that \( X \overset{d}{\rightarrow} \text{Po}(\mu) \) iff (i) \( \sum p_i \rightarrow \mu \) and (ii) \( \max p_i \rightarrow 0 \).
31. In this problem $G = G^{(n)}$ is an $n$-vertex, $d$-regular graph ($d = d(n)$) and $H = H^{(n)}$ is the random subgraph gotten by including each edge of $G$ with probability $p$ ($= p(n)$), these choices made independently (that is, $H$ is the result of percolation on $G$ at probability $p$).

Suppose $d = \omega(1)$ and $(1 - p)^d = \mu/n$, with $\mu > 0$ fixed. If $X = X^{(n)}$ is the number of isolated vertices in $H$, is it true that $X \xrightarrow{d} \text{Po}(\mu)$?

[Caution: watch out for $n$ large relative to $d$.]

32.(a) Let $p = p(n) = (\ln n + c)/n$ with $c$ fixed, and let $G = G^{(n)}$ the random subgraph of $K = K_{n,n}$ gotten by including each edge with probability $p$, independent of other choices (so again, the output of percolation on $K$).

Show

$$\Pr(G \text{ has a perfect matching}) \to e^{-2e^{-e}} \quad (n \to \infty).$$

(b) Similar, a little tougher: with $p$ as above and $n$ even,

$$\Pr(G_{n,p} \text{ has a perfect matching}) \to e^{-e^{-e}} \quad (n \to \infty).$$

[Obvious hints: use Hall’s (Marriage) Theorem for (a) and Tutte’s for (b)—but carefully, or you’re likely to miss some subtleties.]
As promised to a couple of you, answers (not derivations) for Problem 24:

\[ E_X \sim c e^{-2c} n/2 \]
\[ \text{Var} X \sim \left[ \frac{1}{2} c e^{-2c} - c^2 e^{-4c} + c^3 e^{-4c} \right] n. \]

33. For any (locally finite, connected) graph \( G \) and \( v \in V(G) \), define
\[ p_v(G) = \inf \{ p : \mathbb{E}_p |C_v| = \infty \}, \]
(Exercise: this doesn’t depend on \( v \).) Show that \( p_v(G) \leq p_c(G) \) (this is trivial) and that the inequality is sometimes strict.

34. Say a tree \( T \) rooted at \( v \) is spherically symmetric if \( \text{Aut}(T) \) (where automorphisms are required to fix \( v \)) acts transitively on \( V_n \) for each \( n \) (equivalently, if all vertices at any given level have the same degree). Show that for any spherically symmetric \( T \), \( p_c(T) = \limsup |V_n|^{-1/n} \).

35. Give an example of a spherically symmetric tree \( T \) with \( p_c(T) < 1 \) and \( \theta(p_c(T)) \neq 0 \).

36. (Open) As mentioned in class, Lyons’ theorem fails for general graphs. Very roughly, this is shown by providing easy connections within some hard-to-reach cutsets \( \Pi \), so that in the unlikely event that we do reach \( \Pi \), \( |C_v \cap \Pi| \) tends to be large. This suggests the following possible substitute.

Let \( G \) be a (locally finite, connected) graph with “root” \( v \). For a cutset \( \Pi \) and \( w \in \Pi \), let \( A(w, \Pi) \) be the event that there is an open \((v,w)\)-path not meeting \( \Pi \setminus \{w\} \), and set
\[ p'_\text{cut}(G) := \sup \{ p : \inf \sum_{w \in \Pi} \mathbb{P}_p(A(w, \Pi)) = 0 \}. \]
Easy exercise: \( p_{\text{cut}}(G) \leq p'_\text{cut}(G) \leq p_c(G) \) (for any \( G \)).

Question: Could it be that \( p'_\text{cut}(G) = p_c(G) \) for all \( G \)?
It seems sensible to guess that the answer is no; but a counterexample would also be interesting (and publishable).