Remark on 1-3: nothing laborious here, but each needs some little idea.

1. An important though easy fact: for any graph $G$ on vertex set $V$, there’s a partition $X \cup Y$ of $V$ such that

$$|\nabla_G(X,Y)| \geq |G|/2$$

(where $\nabla_G(X,Y)$ is the set of edges of $G$ with one end in each of $X,Y$ and—here and below—we regard $G$ as a set of edges).

Proof. For a random (uniform) partition $X \cup Y$ of $V$, $|G|/2$ is the expected size of $\nabla_G(X,Y)$. □

[Here’s the proof in a little more detail, partly to set notation for possible use in the actual problem: Let $X \cup Y$ be a uniform partition of $V$ (you can say $X$ is a uniform subset of $V$ and $Y = V \setminus X$) and for $e \in G$ let $Z_e = 1_{\{e \in \nabla(G,Z,Y)\}}$. Then $Z := \sum_{e \in G} Z_e = |\nabla_G(X,Y)|$, $EZ_e = 1/2$ and $EZ = |G|/2$ (etc.).]

And finally the problem: show that if $G, H$ are two graphs on $V$ and $\min\{|G|, |H|\}$ is sufficiently large, then there is a partition $V = X \cup Y$ such that

$$|\nabla_G(X,Y)| \geq .49|G| \quad \text{and} \quad |\nabla_H(X,Y)| \geq .49|H|.$$

2. Show there is a constant $C$ such that: if $H$ is a $t$-uniform, $t$-regular hypergraph on $V = [n]$, with $n$ even, then there exists $f : V \to \{\pm 1\}$ with

$$|f(H)| \leq C \sqrt{t \ln t} \quad \forall H \in \mathcal{H}$$

and

$$f(V) = 0.$$

[Easy once found ... ]

3. [You can regard (a) as a bonus; it looks obvious but as far as I know is trickier than it looks (please show I’m wrong). At any rate, feel free to use it for (b).]

(a) Let $X_1, \ldots, X_s$ be chosen uniformly and independently from $[m]$ and set $X = \{X_1, \ldots, X_s\}$. Then $\Pr(X = [m]) \leq [1 - (1 - 1/m)^s]^m$.

(b) (We’ve already seen list-coloring in PS3.4, but now with definitions:)

For a graph $G = (V, E)$ and $S = (S(v) : v \in V)$ with each $S(v)$ a subset of some universe $\Gamma$, a coloring $\sigma : V \to \Gamma$ is $S$-legal if it is proper in the usual
sense and has \( \sigma(v) \in S(v) \) for every \( v \in V \). The list-chromatic number (also called choosability), \( \chi_l(G) \), of \( G \) is the least \( t \) such that for every choice of \( S \) as above with \( |S(v)| = t \ \forall v \), there exists an \( S \)-legal coloring. (So briefly: the least \( t \) such that whenever we’re given \( t \) allowed colors for each vertex, we can find a legal coloring.)

[Little exercise—not to be handed in, just for orientation: show that there are graphs \( G \) with \( \chi_l(G) > \chi(G) \).]

Show that for a bipartite \( G \) of maximum degree \( D \), \( \chi_l(G) = O(D/(\log D)) \).

[Remark. In the background is a particularly infuriating open problem, namely: what (roughly) can one really say here? As far as we know, the right bound could be \( O(\log D) \), but any improvement on the bound in the problem would be very interesting.]

4. Show that if \( n > 2k \) and \( \mathcal{F} \subseteq {\binom{n}{k}} =: \mathcal{K} \) is intersecting of size \( \binom{n-1}{k-1} \), then \( \mathcal{F} = \{ A \in \mathcal{K} : i \in A \} \) for some \( i \in [n] \).

5. Notation: \( Q^n = \{0,1\}^n \) and \( Q^A = \{0,1\}^A \) for \( A \subseteq [n] \). Fix a hypergraph \( \mathcal{H} \) on \( [n] \) and, for any \( \varepsilon = (\varepsilon^A : A \in \mathcal{H}) \) with \( \varepsilon^A \in Q^A \), set

\[
\mathcal{F}(\varepsilon) = \{ x \in Q^n : \exists A \in \mathcal{H}, x_i = \varepsilon^A_i \ \forall i \in A \}.
\]

(Each \( \{ x : x_i = \varepsilon^A_i \ \forall i \in A \} \) is a subcube based on \( A \) and \( \mathcal{F}(\varepsilon) \) is the union of a set of such subcubes, one based on each \( A \in \mathcal{H} \).) Show that \( |\mathcal{F}(\varepsilon)| \) is minimum when \( \varepsilon^A = 0^A \) (i.e. \( \varepsilon^A_i = 0 \ \forall i \in A \) \( \forall A \in \mathcal{H} \).

[Hint: use some kind of shift. Possibly helpful notation: For \( x \in Q^n \) define \( x' \in Q^n \) by \( x'_j = x_j \) iff \( j \neq i \). Write \( x \sim \varepsilon^A \) if \( x_i = \varepsilon^A_i \ \forall i \in A \).]

6. For \( 0 \leq i \leq j \leq n - i \) (which implies \( \binom{n}{i} \leq \binom{n}{j} \)), show that \( M = M(i,j) \) has rank \( \binom{n}{i} \) (i.e. its rows are linearly independent).

[As in class, \( M \) is given by \( M_{A,B} = 1_{\{A \subseteq B\}} \).

Hint postponed to last page in case you’d like to try without it. You might want to use (say) \( \binom{n}{i} = I, \binom{n}{j} = J \).]

7. Derive the EKR Theorem (namely, \( n \geq 2k \) and \( \mathcal{F} \subseteq \binom{[n]}{k} \) intersecting imply \( |\mathcal{F}| \leq \binom{n}{k-1} \)) from the KK Theorem.

8. Recall Turán’s Theorem gives the maximum possible size of a \( K_{r+1} \)-free \( G \) on \( [n] \). Here we stick to \( r|n \), in which case (check) the theorem says

\[
|G| \leq \frac{r-1}{2r} n^2.
\]
Give (two) proofs of this \textit{via} the following approaches.

(a) Show that if $G$ is $K_{r+1}$-free on $[n]$ and $x = (x_1, \ldots, x_n)$ has $x_i \geq 0$ and $\sum x_i = 1$ (i.e. the $x_i$’s are are convex coefficients), then

$$\lambda(x) := \sum_{ij \in E(G)} x_ix_j \leq \frac{r-1}{2r}.$$ 

(b) Show that in any $H$ on $[n]$ with average degree $d$, the expected size of a \textit{random greedy} independent set (say $I$) is at least $n/(d+1)$

[“Random greedy” means we consider the vertices in random order and add $v$ to $I$ if it has no neighbors in $I$. Note $H$ won’t be $G$.]

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\textbf{Hint for problem 6:} (In what follows $I, I'$ always belong to $\mathcal{I}$ and and $J$ is always in $\mathcal{J}$.) Let $\{e_I : I \in \mathcal{I}\}$ be the standard basis for $\mathbb{R}^\mathcal{I}$ and set $f_J = \sum_{I \leq J} e_I$. It’s ETS (why?)

the $f_J$’s span the $e_I$’s.

For this you might consider, for a fixed $I$ and $0 \leq l \leq i$, the vectors

$v_l = \sum_{|J \cap I| = l} f_J$ \quad and \quad $z_l = \sum_{|J \cap I| = l} e_I$.

(Final remarks: (i) this can be done without any actual calculations; (ii) if you get it, where did you use the assumptions on $i$ and $j$?)