1.(a) Let $A_1, \ldots, A_m \in \binom{V}{a}$ and $B_1, \ldots, B_m \in \binom{V}{b}$ satisfy $A_i \cap B_i = \emptyset \ \forall i$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset \ \forall i \neq j$. Then $m \leq (a + b)^a a^{-a} b^{-b}$.

(b) Let $N = [n], \emptyset \neq A \subseteq 2^N \setminus \{\emptyset\}$, and write $\langle A \rangle$ for the filter generated by $A$; that is, $\langle A \rangle = \{B \subseteq N : \exists A \in A, A \subseteq B\}$. For $X \in \langle A \rangle$, let $Z(X) = \bigcap\{A \subseteq X \cap A \in A\}$.

Show that
$$\sum_{X \in \langle A \rangle} |Z(X)(|X| - 1))(n - |X|)! = n!$$

2. Derive Dilworth’s Theorem from König’s.

3.(a) Let $G = (X \cup Y, E)$ be a bigraph, $\alpha : X \to \mathbb{N}$ and $\beta : Y \to \mathbb{N}$. Show that there is some multiset $F$ of $E$ (regarded as the edge set of a multigraph) with
$$d_F(x) = \alpha(x) \ \forall x \in X \ \text{and} \ d_F(y) \leq \beta(y) \ \forall y \in Y$$
if and only if
$$\beta(N(X')) \geq \alpha(X') \ \forall X' \subseteq X, \tag{1}$$
where $d_F(v)$ is the $F$-degree of $v$ (the number of edges of $F$ containing $v$) and (e.g.) $\alpha(S) = \sum_{x \in S} \alpha(x)$ for $S \subseteq X$.

[You don’t have to show the trivial “only if,” but of course should make sure you see why it’s true. Please do not use the max-flow-min-cut theorem.]

(b) Let $G = (V, E)$ be a (general) graph and $k : V \to \mathbb{N}$. Show that $G$ has an orientation satisfying $d^+(v) \leq k(v) \ \forall v \in V$ iff
$$|E(W)| \leq \sum \{k(v) : v \in W\} \ \forall W \subseteq V.$$

[Again, you should understand “only if” but don’t need to write it.]

[Natural definitions: $E(W)$ is the set of edges (of $G$) contained in $W$; an orientation of $G$ orients (or directs) each edge from one of its ends to the other; thus edge $\{x, y\}$ becomes either $xy$ (oriented from $x$ to $y$) or $yx$; an oriented edge (a.k.a. arc) $xy$ has tail $x$ and head $y$; the out-degree, $d^+(x)$, is the number of arcs with tail $x$ (and of course in-degree is defined similarly).]
4. Let $G$ be an $n$-vertex bigraph (say with bipartition $X \cup Y$) and for each $v \in V(G)$ let $S(v)$ be a subset of $\Gamma (= \{\text{"colors"}\})$ of size greater than $\log_2 n$. Then $G$ admits a proper (vertex) coloring $\sigma$ with $\sigma(v) \in S(v) \ \forall v$.

[A-S, 2.7.9]

5. Use a probabilistic argument to show that if $T$ is an $n$-leaf tree in which no node has more than $r$ children, then the average depth of a leaf is at least $\log r n$. (As usual depth is distance from the root.)

6. (For a graph $G$, let $G_p$ be the random subgraph gotten by keeping edges independently, each with probability $p$; e.g. when $G = K_n$, $G_p = G_{n,p}$.) Show that there is a fixed $c > 0$ ($c = 1/2$ will do) for which: if $G = (V, E)$, $|V| = n$ and $\chi(G) = \chi$, then for $H = G_{1/2}$ (and $\log = \log_2$),

$$\Pr(\chi(H) < c\chi/ \log n) = o(1).$$

(This one might be relatively challenging (and interesting). Try to show that $\chi(H) < c\chi/ \log n$ implies some other unlikely event. You may need:

**Proposition.** For any graph $G$ with chromatic number $\chi$, there is some $W \subseteq V(G)$ with $\delta(G[W]) \geq \chi - 1$ (where $\delta$ is minimum degree).

(Note the $G$ here may not be our original $G$. Proving the proposition is also a nice exercise if you haven’t seen it, but not part of the problem.)]

7. Let $S$ be a finite set and $m \in \mathbb{P}$, and consider two experiments:

(a) $x_1, \ldots, x_{2m}$ are chosen uniformly and independently from $S$;
(b) $\{y_1, z_1\}, \ldots, \{y_m, z_m\}$ are chosen uniformly and independently from $\binom{S}{2}$.

Set $X = \{x_1, \ldots, x_{2m}\}$ and $Y = \{y_1, \ldots, y_m, z_1, \ldots, z_m\}$, and prove the rather obvious fact that $\Pr(|Y| \geq t) \geq \Pr(|X| \geq t) \ \forall t$.

8. Fix $c > 0$ and $p = c/n$, and let $X$ be the number of isolated edges in $G_{n,p}$. (An edge is isolated if it meets no others.) Note that for $I$ the (random) set of isolated edges and $e \in E(K_n)$, the probability that $e$ lies in $I$ is $p(1-p)^{2n-4} = \frac{c}{n} \left(1 - \frac{c}{n}\right)^{2n-4} \sim \frac{c}{n} e^{-2c}$; so $EX \sim (\binom{n}{2}) \frac{c}{n} e^{-2c} \sim \frac{cn}{2e^2}$. Give asymptotics for the variance $\sigma^2 = \sigma^2_X$.

[Warning: not hard, but be careful not to discard terms too soon.]

9. Let $p$ be prime, $t \in (\sqrt{p}, p - \sqrt{p})$, and suppose $X, Y$ are $t$-subsets of $\mathbb{Z}_p$. Show that there are $a \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$ with

$$|| (aX + b) \cap Y | - t^2/p | < t/\sqrt{p}$$

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(where $aX + b = \{ax + b : x \in X\}$).

[Hint: Let $a, b$ be chosen uniformly and independently (from $\mathbb{Z}_p^\times$ and $\mathbb{Z}_p$ resp.), let $\varphi(x) = ax + b$ ($x \in \mathbb{Z}_p$), and consider the distribution of $(\varphi(x), \varphi(y))$ when $x, y$ are distinct elements of $\mathbb{Z}_p$.]