1. Use the generating function \( \prod_{j \geq 0} (1 + x^{2^j})^{-1} \) to show that for \( n \geq 2 \), the number of partitions of \( n \) into powers of 2 is even. 

[E.g. for \( n = 4 \), the relevant partitions are 4, 22, 211 and 1111.]

2. Let \( p'(n) \) (resp. \( p''(n) \)) be the number of partitions of \( n \) into an even (resp. odd) number of parts, and \( t(n) \) the number of partitions of \( n \) into distinct odd parts. Show \( |p'(n) - p''(n)| = t(n) \).

3. Let \( a_n \) be the number of involutions (permutations \( \sigma \) with \( \sigma^2 = \text{id} \)) of \( [n] \) (with \( a_0 := 1 \)).
   
   (a) Use inclusion-exclusion to find a closed form for the sum
   \[
   \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a_i.
   \]

   (b) Find a closed form for the exponential generating function, say \( f(x) \), of the sequence \( \{a_n\} \). [Preferred solution: get this from (a).]

4. For a partition \( \lambda \), let \( u(\lambda) \) and \( v(\lambda) \) be (respectively) the number of times \( 1 \) appears as a part of \( \lambda \) and the number of different integers that appear as parts of \( \lambda \). Use 2-variable generating functions to show that (for any \( n \))
   \[
   \sum_{\lambda \vdash n} u(\lambda) = \sum_{\lambda \vdash n} v(\lambda).
   \]

   [It can also be done “combinatorially,” but that’s not what’s asked for here.]

5. Suppose \( 2n \) people, named \( X_1, Y_1, \ldots, X_n, Y_n \), are seated (uniformly) at random around a circular table. Give (and justify) asymptotics for the probability that no \( X_i, Y_i \) are seated next to each other.

   [You can think of a random cyclic ordering, instead of a random assignment to fixed seats; this is equivalent of course, but may be easier to work with.]

6. Let \( V = V_1 \cup \cdots \cup V_k \) be a partition with \( |V_i| = n \ \forall i \), and say \( T \in \binom{V}{k} \) is a transversal if it meets every \( V_i \). Show that if \( h : \binom{V}{k} \to \mathbb{R} \) satisfies \( h(T) = 1 \) for each transversal \( T \), then there is some \( S \subseteq V \) with
   \[
   |h(S)| \geq c_k n^k,
   \]
where $c_k > 0$ depends only on $k$ and $\overline{h}(S) = \sum \{h(T) : T \subseteq S, |T| = k\}$.  

[For $X \subseteq \binom{V}{k}$ please use $h(X) = \sum_{E \subseteq X} h(E)$.

7. (a) Suppose $A_1, \ldots, A_m, B_1, \ldots, B_m \subseteq X$ satisfy

$$|A_I| = |B_I| \quad \forall I \subset [m] \quad \text{and} \quad |A_m| \neq |B_m|$$

(where $A_I = \cap_{i \in I} A_i$ and similarly for $B_I$). Show that $|X| \geq 2^{m-1}$.

(b) The bound in (a) is best possible (for every $m$).

[It may help to think of the Venn diagrams associated with the $A$’s and $B$’s.]

8. Let $A(P)$ be the poset whose elements are the antichains of the poset $P$, with $A \leq B$ iff for each $a \in A$ there’s some $b \in B$ with $b \geq a$. Show that $A(P)$ is (isomorphic to) a subposet of $2^P$.

9. Show that there is a fixed $C > 1$ such that $|\text{End}(P)| > C^n$ for each $n > 1$ and poset $P$ of size $n$ (i.e. with ground set of size $n$).

[For simplicity let’s restrict to $P$’s in which each element is comparable to at least one other. (Of course “isolated” elements just make it easier, right?)]