1. Show that geometric lattices are coatomic (i.e. have the property that every element is a meet of coatoms).

2. For \( \mathcal{H} \) as in the EFL Conjecture and \( n \geq 3 \), show \( \chi'(\mathcal{H}) \leq 2n - 3 \).
   [Familiar pseudohint: easy once found.]

3. Use Motzkin’s Lemma to prove:
   (a) If \( \mathcal{H} \) is a hypergraph such that \( |A \cap B| = 1 \) for all distinct \( A, B \in \mathcal{H} \), then \( |\mathcal{H}| \leq \Delta(\mathcal{H}) \), where
      \[
      \Delta(\mathcal{H}) = \max\{ |\cup_{x \in A \in \mathcal{H}} A| : x \in V(\mathcal{H}) \}.
      \]
   [Let’s say \( N(x) = \bigcup\{ A : x \in A \in \mathcal{H} \} \ (x \in V(\mathcal{H})) \). Cryptic hint: the most natural way to start this one doesn’t work (as far as I know).]
   [The coloring version is also conjectured: if intersection sizes in \( \mathcal{H} \) are at most 1, then \( \chi'(\mathcal{H}) \leq \Delta(\mathcal{H}) \); note that this strengthening of the EFL Conjecture would also generalize Vizing’s Theorem.]
   (b) If \( L \) is a geometric lattice of rank \( r \), then \( W_1 \leq W_{r-1} \).

   [You may use without proof (though you should see why it’s true) the fact that intervals of geometric lattices are geometric, an interval of a poset \( P \) being \( [x, y] = \{ z : x \leq z \leq y \} \) for some \( x, y \in P \) with \( x \leq y \) (more precisely, this set equipped with the relations inherited from \( P \)). Please use copoint and coline for lattice elements of ranks \( r-1 \) and \( r-2 \), \( a(x) \) for the number of copoints above \( x \in L \), and \( h \) for the number of points (atoms) below \( h \). You can skip fiddling with the pathological exceptions in Motzkin’s lemma.]

4. For a lattice \( L \) with set of atoms \( A \),
   \[
   \mu_L(\hat{0}, \hat{1}) = \sum\{ (-1)^{|S|} : S \subseteq A, \forall S = \hat{1} \}.
   \]

5. For \( L = \Pi_n \) (the partition lattice), find \( \mu_L(\hat{0}, \hat{1}) \) using:
   (a) Weisner,
   (b) dual Weisner.
6. Let \( L \) be a geometric lattice of rank \( r \), \( k < r = 2 \), and

\[ B_k = \{ x : r(x) \leq k \}, \quad T_k = \{ x : r(x) \geq r - k \}. \]

Show that there is an injection \( f : B_k \to T_k \) with \( f(x) \geq x \) for all \( x \in B_k \).

[Hint: Take \( \{ e_x : x \in L \} \) to be the standard basis for \( \mathbb{R}^L \), and for \( y \in T_k \) set \( w_y = \sum \{ e_x : x \in B_k, x \leq y \} \). Let \( M \) be the \( B_k \times T_k \) matrix whose columns are the vectors \( w_y \), i.e.

\[ M(x, y) = \begin{cases} 1 & x \leq y \\ 0 & \text{otherwise} \end{cases} \]

(a piece of the \( \zeta \)-matrix.) Show that \( \text{rank}(M) = |B_k| \), i.e. that the vectors \( w_y \) span \( \mathbb{R}^{B_k} \) (and explain why this gives the statement in the problem). It may help to also consider the vectors \( v_y = \sum \{ e_x : x \in B_k, x \& y = 1 \} \) (again, for \( y \in T_k \)). This one can be tricky, though it’s not hard once you find it.]

7. Suppose \( X \subseteq \mathbb{Z}_3^n \) has the property that for all distinct \( x, y \in X \) there is some \( i \) for which \( y_i = x_i + 1 \) (addition in \( \mathbb{Z}_3 \) of course). Then \( |X| \leq 2^n \).

[More general (not required but you could try): Let \( q \) be a prime power and \( D \) a \( d \)-subset of \( \mathbb{F}^n_q \), and suppose \( X \subseteq \mathbb{F}^n_q \) satisfies: for all distinct \( x, y \in X \) there is some \( i \) for which \( y_i - x_i \in D \). Then \( |X| \leq (d + 1)^n \).]

8. For \( X \subseteq \mathbb{Z}_n \) and \( i \in \mathbb{Z}_n \), let \( X + i = \{ x + i : x \in X \} \), where addition is modulo \( n \). Here are two old conjectures (the first reminiscent of EKR and the Simonovits-Sós conjecture mentioned in class):

**Conjecture 1.** Let \( X \) be a \( k \)-subset of \( \mathbb{Z}_n \) and suppose \( \mathcal{F} \subseteq 2^{\mathbb{Z}_n} \) satisfies

\[ \forall A, B \in \mathcal{F}, \quad A \cap B \supseteq X + i \quad \text{for some } i \in \mathbb{Z}_n. \]

Then \( |\mathcal{F}| \leq 2^{n-k} \).

**Conjecture 2.** For any \( k \)-subset \( X \) of \( \mathbb{Z}_n \), there is a \( k \times n \) \( \{0,1\} \)-matrix \( M \) such that for each \( i \in \mathbb{Z}_n \) the columns of \( M \) indexed by \( X + i \) are linearly independent over \( \mathbb{Z}_2 \).

Show Conjecture 2 implies Conjecture 1. (Full credit, and some additional benefits, for proving Conjecture 1 (with or without using Conjecture 2).)