642.583 Problem Set 1 (final)

[Please see the homework guidelines on the course page.
If something seems wrong, please ask before wasting a lot of time on it.
All problem parts have equal weight.]

1. For a closure \( \lambda \) on a (say finite) set \( X \), the collection of independent sets of \( \lambda \) is
\[
I_\lambda = \{ A \subseteq X : x \notin \lambda(A \setminus x) \forall x \in A \}.
\]
(Of course \( A \setminus x \) should really be \( A \setminus \{x\} \).) Prove or disprove: if \( I \subseteq 2^X \) is an ideal containing \( \emptyset \), then there is a closure \( \lambda \) on \( X \) with \( I_\lambda = I \).

2. Prove that for a lattice \( L \),
\[
(u \lor v) \land w = (u \land w) \lor (v \land w) \quad \forall u, v, w
\]
implies
\[
(x \land y) \lor z = (x \lor z) \land (y \lor z) \quad \forall x, y, z.
\]

3. Let \( L \) be a finite distributive lattice, and for \( k \in \mathbb{N} \) let \( A_k \) (resp. \( B_k \)) be the subposet of \( L \) consisting of those elements of \( L \) which cover (resp. are covered by) exactly \( k \) elements. Show:
(a) \( |A_k| = |B_k| \),
(b) \( A_1 \cong B_1 \), but in general \( A_k \) need not be isomorphic to \( B_k \).

4. For a poset \( P \) and \( a, b \in P \) with \( a \leq b \), we write \( [a, b] \) for the “interval” \( \{ x \in P : a \leq x \leq b \} \).

A lattice \( L \) is meet-distributive if for each \( y \in L \), \( \land_{x \leq y} x, y \) is a Boolean algebra. [Exercise (not to be handed in, but legal to use): this is equivalent to saying that if \( \{ x : x \prec y \} = \{ x_1, \ldots, x_t \} \) and we set \( x_I = \land_{i \in I} x_i \) (for \( I \subseteq [t] \)), then \( [x_I, y] = \{ x_I : I \subseteq [t] \} \) and \( I \neq J \Rightarrow x_I \neq x_J \).]

A closure \( A \mapsto \overline{A} \) is antiexchange if
\[
y \neq x \in \overline{A} \cup y \setminus \overline{A} \quad \Rightarrow \quad y \notin \overline{A} \cup x.
\]
(Motivation: this is an abstraction/generalization of convex closure.)

Show that TFAE by showing (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i) (so a 3-part problem).
(i) \( L \) is meet-distributive;
(ii) \( x \mapsto |S(x)| \) is a rank function;

(iii) The closure associated with \( L \) is antiexchange.

[Small suggestion for the first implication: show \( x < y \Rightarrow |S(y) \setminus S(x)| = 1 \) by induction on \( |S(y)| \). As above, feel free to use \( A \cup x \) for \( A \cup \{x\} \) and, similarly, \( A \setminus x \) for \( A \setminus \{x\} \).
Please try to do this one efficiently (in the eventual posted solutions the whole thing will take well under a page). You might also want to watch out for fooling yourself—make sure your assertions are really justified.]

5. (Recall that we associate with a closure space, \((X, \lambda)\), a collection of independent sets, \( I_\lambda = \{ A \subseteq X : x \notin \lambda(A \setminus x) \quad \forall x \in A \}. \)

(a) If \( \lambda \) is exchange, then \( I = I_\lambda \) satisfies the matroid independence axioms. You can skip the first two (less interesting) axioms and just show
\[
(\text{I3}) \quad \forall A, B \in I \text{ with } |A| < |B|, \ \exists x \in B \setminus A \text{ with } A \cup x \in I.
\]
[Hint: Take \((A, B)\) to be a pair violating (I3) with \( A \setminus B \) minimal.]

(b) If \( \emptyset \neq I \subseteq 2^X \) is an ideal with (I3), then there is an exchange closure \( \lambda \) on \( X \) with \( I_\lambda = I \).

[Possibly helpful notions here are basis and rank: a basis of \( Y \subseteq X \) is a maximal independent subset (of \( Y \)), and, as in class, the rank, \( r(Y) \), is the common size of these bases. The proofs of monotonicity and idempotence of \( \lambda \) (once \( \lambda \) is defined) are similar, so let’s say you should just show idempotence and the exchange property.
Comments following Problem 4 regarding efficient writing and “be careful” apply here as well: once you have some proof, you should see what you can do to streamline and write concisely; this is good practice, if a bit painful. And take heart: we won’t see many—if any—more like this.]