# RESEARCH STATEMENT

LIVIU ILINCA

## 1. Overview

My mathematical interests lie in the general area of discrete mathematics, particularly extremal combinatorics and graph theory, and probabilistic methods. My current research deals with problems related to the asymptotic enumeration of combinatorial objects. I have obtained the following results:

(1) A new proof for a theorem of P.Allen [All07] on the number of 2-SAT functions of $n$ variables. In the process, I have determined the asymptotics for the number of odd-blue-triangle-free graphs, a question naturally arising in the [BBL03] approach to the 2-SAT problem above. (with J. Kahn)
(2) Asymptotics for the number of 3-SAT functions of $n$ variables, proving (in a strong form) a conjecture of Bollobás, Brightwell and Leader [BBL03]. (with J. Kahn)
(3) An upper bound for the number of matchings of size $\ell$ in a graph, which represents the best progress to date on the Upper Matching Conjecture of Friedland, Krop, Lundow and Markström [FKLM08]. (with J. Kahn)

## 2. The $k$-SAT problem

Let $\{x_1, \ldots, x_n\}$ be a collection of Boolean variables. Each variable $x$ is associated with a *positive* literal, $x$, and a *negative* literal, $\bar{x}$. A *k-SAT formula* is an expression of the form

$$\mathscr{C} = C_1 \vee \cdots \vee C_t,$$

with $t$ a positive integer and each $C_i$ a *k-clause*; that is, an expression $y_1 \wedge \cdots \wedge y_k$, with $y_1, \ldots, y_k$ literals corresponding to different variables. A formula $\mathscr{C}$ defines a Boolean function of $x_1, \ldots, x_n$ in the natural way; here we usually think of this as the set $F(\mathscr{C})(\subseteq \{0,1\}^n)$ of satisfying assignments for $\mathscr{C}$; any such function is a *k-SAT function.*

We are interested in the number $G_k(n)$ of k-SAT functions of $n$ variables. Of course, $G_k(n)$ is at most $\exp_2[2^k\binom{n}{k}]$, the number of k-SAT formulas. On the other hand, all formulas obtained by choosing $y_i \in \{x_i, \overline{x}_i\}$ for each $i$ and

---

*Date*: November 15, 2009.

a set of clauses using precisely the literals $y_1, \ldots, y_n$ give different functions, hence[1]

$$G_k(n) > 2^n(2^{\binom{n}{k}} - n2^{\binom{n-1}{k}}) \sim 2^{n+\binom{n}{k}}. \tag{1}$$

The problem of estimating $G_k(n)$ was suggested by Bollobás *et al.* [BBL03]. They showed that

$$\log_2 G_2(n) \sim \binom{n}{2} \tag{2}$$

and, for $k \leq n/2$,

$$G_k(n) \leq \exp_2\left[\sqrt{\pi(k+1)}\binom{n}{k}\right].$$

The natural conjecture is that (1) gives the asymptotic value of $G_k(n)$:

**Conjecture 2.1.** *For each fixed $k$, $G_k(n) = (1+o(1))2^{n+\binom{n}{k}}$.*

Note that even the weaker

$$\log_2 G_k(n) \sim \binom{n}{k},$$

conjectured in [BBL03], was until now open for $k > 2$.

### 2.1. **The number of 2-SAT functions.**

**Theorem 2.2.**

$$G_2(n) = (1+o(1))2^{n+\binom{n}{2}}.$$

This was proved in [All07], and we gave an alternate proof. An interesting feature of our argument is that it follows the original colored graph approach of [BBL03], and is inspired by the wide literature on asymptotic enumeration (for example [EKR76, KR75, PSS02]). Both [All07] and [BBL03] mention the seeming difficulty of proving Theorem 2.2 following this natural approach, and their proofs are instead based on the Regularity Lemma of Szemerédi [Sze78].

2.2. **The number of OBTF graphs.** We consider *colored* graphs, meaning graphs with edges colored *red* ($R$) and *blue* ($B$). For such a graph $G$, a subset of $E(G)$ is *odd-blue* if it has an odd number of *blue* edges (and *even-blue* otherwise), and (of course) $G$ is *odd-blue-triangle-free* (OBTF) if it contains no odd-blue triangle.

The argument of [BBL03] reduces (2) to a question about the number, say $F(n)$, of OBTF graphs. The main part of their argument is devoted to proving

$$\log_2 F(n) \sim \binom{n}{2},$$

from which (2) follows easily.

---

[1] $f(n) \sim g(n)$ means $f(n)/g(n) \to 1$ as $n \to \infty$.

Say a graph $G$(colored as above) is *blue-bipartite* if there exists a partition $U \cup W$ of $V(G)$ such that each blue edge has one endpoint in each of $U$, $W$, while any red edge is contained in one of $U$, $W$. If $B(n)$ is the number of blue-bipartite graphs on $n$ vertices, then it is easy to see that

$$B(n) = (1 - o(1))2^{(n-1)+\binom{n}{2}}$$

(the r.h.s. counts ways of choosing the unordered pair $\{U, W\}$ and an *uncolored* $G$, the coloring then being dictated by "blue-biparticity").

One big step in our proof of Theorem 2.2 is showing the natural conjecture that *most OBTF graphs are blue-bipartite*:

**Theorem 2.3.**

$$F(n) = (1 + o(1))2^{(n-1)+\binom{n}{2}},$$

though in our case, unlike in [BBL03], deriving Theorem 2.2 from this required a fairly difficult argument.

2.3. **The number of 3-SAT functions.** We have proved Conjecture 2.1 for $k = 3$, namely:

**Theorem 2.4.**

$$G_3(n) = (1 + o(1))2^{n+\binom{n}{3}}.$$

The proof of Theorem 2.4 involves, among other things, the Hypergraph Regularity Lemma (HRL) of Frankl and Rödl [FR02], a pioneering extension of the (graph) Regularity Lemma [Sze78] mentioned above, and some entropy considerations, including Shearer's Lemma [CFGS86]. The proof is quite a long story (currently about 40 pages) and does not lend itself to a quick sketch, but I will mention one simple but crucial idea.

Call a ($k$-SAT) formula $\mathscr{C}$ *irredundant* if any proper subformula produces a different function than $F(\mathscr{C})$ (thus, for each clause $C \in \mathscr{C}$, there is some $w \in \{0, 1\}^n$ – a *witness* of $C$ – that satisfies $C$ but no other clause in $\mathscr{C}$). Of course, the number $I_k(n)$ of irredundant formulas (IFs) is at least $G_k(n)$, since each $k$-SAT function is given by at least one IF, and it is easy to see that there are functions that are given by many IFs. Nonetheless, a key insight in the proof of Theorem 2.4 was the guess that $I_3(n)$ *is asymptotically the same as* $G_3(n)$ (which, once shown, says that in fact most functions are given by exactly one IF). Thus what we actually end up proving is the IF version of Theorem 2.4:

**Theorem 2.5.**

$$I_3(n) = (1 + o(1))2^{n+\binom{n}{3}}.$$

Of course, one thing I would like to do in the future is to prove Conjecture 2.1 – or at least the weaker $\log_2 G_k(n) \sim \binom{n}{k}$– for general $k$. It is a little surprising that the proof of Theorem 2.4 does not completely extend to settle this, since experience in many analogous situations suggests that the big jump in difficulty should be that from $k = 2$ to $k = 3$. In fact, much of the current argument does extend (using [Gow07] or [RS04] in place of [FR02]), but there are several *ad hoc* bits at the beginning of the proof whose growth as $k$ increases we have so far been unable to control. (For example, I am pretty sure we could do $k = 4$ if that were the goal, but at this point the interest is really in general $k$).

## 3. THE NUMBER OF MATCHINGS OF A GIVEN SIZE

Given $G = (V, E)$ a graph with $|V| = N$ vertices and $\ell$ an integer with $0 \leq \ell \leq N/2$, let $\Phi_\ell(G)$ be the number of matchings[2] of size $\ell$ of $G$. Friedland *et al.* [FKLM08] proposed the following Upper Matching Conjecture:

**Conjecture 3.1.** *If $G$ is a $d$-regular graph on $N$ vertices with $2d|N$ and $\ell$ is an integer satisfying $0 \leq \ell \leq N/2$ then*

$$\Phi_\ell(G) \leq \Phi_\ell(DK_{N,d}),$$

*where $DK_{N,d}$ is the union of $\frac{N}{2d}$ disjoint copies of the complete bipartite graph $K_{d,d}$.*

A special case of this is when $N$ is even and $\ell = N/2$, thus $\Phi_{N/2}(G)$ is the number of perfect matchings of $G$. In the bipartite case, this is a well-known result of Brégman [Bre73], that was extended to general graphs in [KL04, Ego07, Fri08, AF08].

Friedland, Krop and Markström [FKM08] proved Conjecture 3.1 in the case $\ell = 2$ and Carroll, Galvin and Tetali [CGT09] provided asymptotic evidence for general $\ell$, showing that[3], with $\alpha = 2\ell/N$,

$$\log_2 \Phi_\ell(G) \leq \frac{N}{2}\left[\alpha \log_2 d + H(\alpha)\right]. \tag{3}$$

This contrasts with the obvious lower bound (which Conjecture 3.1 would say is the truth), given by

$$\log_2 \Phi_\ell(DK_{N,d}) \geq \frac{N}{2}\left[\alpha \log_2 d + 2H(\alpha) + \alpha \log_2\left(\frac{\alpha}{e}\right) + \Omega\left(\frac{\log_2 d}{d}\right)\right]. \tag{4}$$

Note that the main interest here is in the discrepancy, $\frac{N}{2}[H(\alpha) + \alpha\log_2\left(\frac{\alpha}{e}\right)]$, between the terms of order $N$ in the upper bound (3) and the lower bound (4). We closed this gap:

**Theorem 3.2.** *Let $G$ be a $d$-regular graph on $N$ vertices. Let $\ell$ be an integer satisfying $0 \leq \ell \leq N/2$. Set $\alpha = 2\ell/N$. Then*

$$\log_2 \Phi_\ell(G) \leq \frac{N}{2}\left[\alpha \log_2 d + 2H(\alpha) + \alpha \log_2\left(\frac{\alpha}{e}\right) + O(d^{-2/3})\right].$$

---

[2] A matching is a set of edges without common vertices.

[3] $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$, for $x \in (0,1)$, is the usual binary entropy function.

Actually, we proved a more general result (whose statement is omitted, for the sake of brevity), namely a similar looking upper bound for $\Phi_\ell(G)$ *given the degree sequence* $\{d_v\}_{v\in V(G)}$. Moreover, if $G$ is bipartite with vertex set $X \cup Y$, all we need for this are the degrees $\{d_x\}_{x\in X}$.

The proofs of these results are mostly based on entropy considerations, in the spirit of Radhakrishnan's proof of Brégman's Theorem [Rad97], and, for example, the more recent [CK09]. Here, as for some related problems, most of the work is devoted to the bipartite case. The passage to general graphs is then accomplished by way of an easy correspondence between ordered pairs of $\ell$-matchings of $G$ and a subset of the $2\ell$-matchings of the bipartite double cover of $G$. (This correspondence, which goes back at least to Gibson [Gib70], was recently rediscovered by Alon and Friedland [AF08].)

I am still trying to prove Conjecture 3.1 in full, though this does not seem easy. On the other hand, I am hoping that the methods used in proving Theorem 3.2 can be applied in other contexts, for example for the following

**Conjecture 3.3.** *For fixed $\Delta$ and $T$ an $n$-vertices tree of maximum degree at most $\Delta$, the number $c(T)$ of copies of $T$ in an $(n/2)$-regular graph $G$ on $n$ vertices can not be significantly larger than what we expect to see in the random graph $G(n,1/2)$. For example, if $T_0$ is the complete binary tree on $n = 2^k - 1$ vertices, then*

$$\log_2 c(T_0) \le n\left[\log_2\left(\frac{n}{e}\right) - \frac{3}{2} + o(1)\right].$$

## REFERENCES

[All07] P. Allen, Almost every 2-SAT function is unate, *Israel J. Math.* **161** (2007), 311–346.

[AF08] N. Alon and S. Friedland The maximum number of perfect matchings in graphs with a given degree sequence, *Electron. J. Combin.* **15** (2008), N13.

[BBL03] B. Bollobás, G. Brightwell and I. Leader, The number of 2-SAT functions, *Israel J. Math.* **133** (2003), 45–60.

[Bre73] L. Brégman, Some properties of nonnegative matrices and their permanents, *Math. Doklady* **14** (1973), 945–949.

[CFGS86] F.R.K. Chung, P. Frankl, R. Graham and J.B. Shearer, Some intersection theorems for ordered sets and graphs, *J. Combinatorial Th. Ser. A.* **48** (1986), 23–37.

[CGT09] T. Carroll, D. Galvin and P. Tetali, Matchings and independent sets of a fixed size in regular graphs, *J. Combin. Theory Ser. A* **116** (2009), no. 7, 1219–1227.

[CK09] B. Cuckler and J. Kahn, Entropy bounds for perfect matchings and Hamiltonian cycles, *Combinatorica* **29** (2009), no. 3, 327-335.

[EKR76] P. Erdős, D.J. Kleitman and B.L. Rothschild, Asymptotic enumeration of $K_n$-free graphs, *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973)* Tomo II, 19-27. Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976.

[FR02] P. Frankl, V. Rödl, Extremal Problems on Set Systems, *Random Structures and Algorithms* **20** (2002), no. 2, 131-164.

[Fri08] S. Friedland, An upper bound for the number of perfect matchings in graphs, arXiv:0803.0864.

[FKLM08] S. Friedland, E. Krop, P. Lundow, and K. Markström, On the validations of the asymptotic matching conjectures *J. Stat. Phys.* **133** (2008), no. 3, 513–533.

[FKM08] S. Friedland, E. Krop and K. Markström, On the Number of Matchings in Regular Graphs, *The Electronic Journal of Combinatorics* **15**(1) (2008), R110.

[Ego07] G.P. Egorychev, *Permanents*, Book in Series of Discrete Mathematics (in Russian), Krasnoyarsk, SFU, 2007.

[Gib70] P.M. Gibson, Combinatorial matrix functions and 1-factors of graphs. *SIAM J. Appl. Math.* **19** (1970), 330–333.

[Gow07] W. T. Gowers, Hypergraph regularity and the multidimensional Szemerdi theorem, *Ann. of Math.*(2) **166** (2007), no. 3, 897–946.

[IK1] L. Ilinca and J. Kahn, The Number of 3-SAT Functions, in preparation.

[IK2] L. Ilinca and J. Kahn, The Number of Matchings of a Given Size, preprint available

[IK3] L. Ilinca and J. Kahn, On the Number of 2-SAT Functions, in *Combinatorics, Probability and Computing*, **18**(2009), no. 5, 749–764.

[KL04] J. Kahn, L. Lovász, unpublished, 2004.

[KR75] D.J. Kleitman and B.L. Rothschild, Asymptotic enumeration of partial orders on a finite set, *Trans. Amer. Math. Soc.* **205** (1975), 205–220.

[PSS02] H.J. Prömel, T.Schickinger and A. Steger, A note on triangle-free and bipartite graphs, *Disc. Math.* **257** (2002), 531–540.

[Rad97] J. Radhakrishnan, An Entropy Proof of Bregman's Theorem, *J. Combin. Theory Ser. A* **77** (1997), 161–164.

[RS04] V. Rödl and J. Skokan, Regularity lemma for k-uniform hypergraphs, *Random Structures Algorithms*, **25** (2004), no. 1, 1–42.

[Sch78] A. Schrijver, A short proof of Minc's conjecture, *J. Combin. Theory Ser. A* **25** (1978), no.1, 80–83.

[Sze78] E. Szemerédi, Regular Partitions of Graphs, pp. 399–401 in *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, Paris: Éditions du Centre National de la Recherche Scientifique (CNRS), 1978.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854

*E-mail address*: `ilinca@math.rutgers.edu`