

## Midterm Review

1. Define the following terms:
  - (a) (general) graph, simple graph
  - (b) isomorphic graphs
  - (c) complete graph, regular graph, bipartite graph
  - (d) walk, trail, path, cycle in a graph
  - (e) connected graph
  - (f) cutset, bridge; edge connectivity  $\lambda(G)$ ,  $k$ -edge-connected graph
  - (g) separating set, cut-vertex; connectivity  $\kappa(G)$ ,  $k$ -connected graph
  - (h) Eulerian trail; Eulerian graph, semi-Eulerian graph
  - (i) Hamiltonian cycle; Hamiltonian graph, semi-Hamiltonian graph
  - (j) forest, tree; spanning tree, spanning forest
  - (k) planar graph
  - (l) geometric dual of a graph; abstract dual of a graph
  - (m) infinite graph, countable graph, locally finite graph
2. State the following theorems:
  - (a) The Handshaking Lemma
  - (b) Euler's Theorem (for Eulerian graphs)
  - (c) Fleury's Algorithm
  - (d) Ore's Theorem
  - (e) Dirac's Theorem
  - (f) Equivalent Definitions of a Tree
  - (g) Cayley's Theorem
  - (h) The Greedy Algorithm for finding a *minimum-weight* spanning tree
  - (i) Kuratowski's Theorem
  - (j) Euler's Formula (for planar graphs)

3. Are two given graphs isomorphic? (e.g. Exercise 2.3)
4. Draw all non-isomorphic unlabelled graphs on 3 or 4 vertices.
5. Given the degree sequence  $(3, 3, 5, 5, 5, 5)$ , does there exist a *simple* graph with these degrees?
6. Draw a graph given its adjacency or incidence matrix. (e.g. Exercise 2.13)
7. Draw, say how many vertices and edges are there and note if they are regular or bipartite, for the following graphs:  $K_n$ ,  $K_{n,m}$ ,  $W_n$ ,  $P_n$ ,  $C_n$ , the Petersen graph, the tetrahedron, the octahedron, the cube.
8. Exercise 3.8:
  - (a) Prove that if  $G$  is self-complementary (isomorphic to its complement), then the number of vertices is of the form  $4k$  or  $4k + 1$ , where  $k$  is an integer.
  - (b) Find all self-complementary graphs on 4 and 5 vertices.
9. If  $G$  is a bipartite graph, then any cycle has even length.
10. Exercise 5.1: In the Petersen graph, find cycles of lengths 5, 6, 8 and 9; cutsets with 3, 4 and 5 edges.
11. Find  $\kappa(G)$  and  $\lambda(G)$  for some graphs (e.g. Exercise 5.5)
12. Draw a graph  $G$  with minimum degree  $d$  such that
$$\kappa(G) < \lambda(G) < d$$
13. Exercise 5.9.
14. Exercise 5.10.
15. Show that if  $G$  is an Eulerian graph, then all degrees are even.
16. Use Euler's Theorem to show that a connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.
17. Given a graph, say if it is Eulerian or semi-Eulerian (e.g. Exercises 6.1 and 6.3).

18. Show that Ore's Theorem implies Dirac's Theorem
19. Given a graph, say if it is Hamiltonian or semi-Hamiltonian (e.g. Exercises 7.1 and 7.3)
20. Exercise 7.6: Give an example to show that the condition ' $\deg(v) \geq n/2$ ' in the statement of Dirac's Theorem cannot be replaced by

$$\deg(v) \geq (n - 1)/2$$

21. Use Ore's Theorem to show that, if  $G$  is a graph with  $n$  vertices and  $(n - 1)(n - 2)/2 + 2$  edges, then  $G$  is Hamiltonian.
22. Solve a shortest path problem. (e.g. Exercise 8.4)
23. Solve a **longest** path problem. (e.g. Exercise 8.3)
24. Solve a Chinese Postman Problem, for some semi-Euclidean weighted graph (e.g. Exercise 8.5)
25. Prove Theorem 9.1.
26. From Theorem 9.1 we have that any tree on  $n$  vertices has exactly  $n - 1$  edges.
  - (a) Prove that any forest with  $n$  vertices and  $k$  connected components has exactly  $n - k$  edges.
  - (b) Prove that any graph with  $n$  vertices and  $k$  connected components has at least  $n - k$  edges.
27.
  - (a) Prove that any tree has at least 2 leaves.
  - (b) What trees have exactly 2 leaves?
28. Prove Theorem 9.3.i: If  $G$  is a graph,  $F$  is a spanning forest in the graph  $G$  and  $C$  is a cutset of the graph  $G$ , then:

$$F \cap C = \emptyset$$

29. Exercise 9.2: Show, by drawing, that there are (up to isomorphism) six trees on 6 vertices.

30. Exercise 9.3:
  - (a) Prove that every tree is a bipartite graph.
  - (b) Which trees are complete bipartite graphs?
31. Exercise 9.4: Draw all the spanning trees in the graph of Figure 9.5
32. Exercise 10.4: How many spanning trees has the graph  $K_{2,n}$ ?
33. Show that Cayley's Theorem holds for  $n=4$  (See the beginning of Section 10).
34. Find the number of trees on  $n$  vertices in which a given vertex  $v$  has degree:
  - (a)  $\deg(v) = 1$ .
  - (b)  $\deg(v) = n - 2$ .
  - (c)  $\deg(v) = n - 3$ .
35. Theorem 11.1: Prove that the *greedy* algorithm does produce a *minimum-weight* spanning tree.
36. Find a *maximum-weight* spanning tree in the graph of Figure 11.1.
37. Give an example of a graph  $G$  in which the *greedy* algorithm can produce two distinct *minimum-weight* spanning trees.
38. Theorem 12.1:  $K_5$  and  $K_{3,3}$  are not planar.
39. Given some graph, show whether it is planar or not. (e.g. Exercise 12.2).
40. Which complete graphs and complete bipartite graphs are planar?
41. Exercise 12.6: Prove that the Petersen graph is non-planar.
42. Prove Euler's Formula and use it to show that  $K_5$  and  $K_{3,3}$  are non-planar.
43. (a) If  $G$  is a connected simple planar graphs with no triangles, then  $m \leq 2n - 4$ . (i.e. Corrolary 13.4.ii).

- (b) Same question as in part (a), but now  $G$  contains no cycles of length less than 5.
- (c) Use part (b) to show that the Petersen graph is non-planar.
44. Show that every simple planar graph contains a vertex of degree at most 5. (i.e. Theorem 13.6).
45. Given some plane drawing of a graph  $G$ , redraw the graph so that a given face becomes the infinite face. (e.g. Exercise 13.2).
46. Show that if  $G$  has at least 11 vertices, then  $G$  and  $\bar{G}$  cannot both be planar.
47. By Theorem 14.2, we know that if  $G$  is a connected graph of genus  $g$  with  $n$  vertices,  $m$  edges and  $f$  faces, then:
- $$n - m + f = 2 - 2g$$
- (a) Prove Corollary 14.3:  $g \geq (m - 3n)/6 + 1$
- (b) Use (a) to prove the lower bound in the Ringel-Youngs Theorem, that is:
- $$g(K_n) \geq (n - 3)(n - 4)/12$$
- (c) Exercise 14.3.i: Calculate  $g(K_7)$  and  $g(K_{11})$ .
48. Exercise 14.5.
49. Theorem 15.3: Let  $G$  be a planar graph and  $G^*$  be a geometric dual of  $G$ . If a set of edges in  $G$  forms a cycle in  $G$ , then the corresponding set of edges of  $G^*$  forms a cutset in  $G^*$ .
50. Find some duals:
- (a) Exercise 15.2: the dual of the cube graph is the octahedron graph.
- (b) Exercise 15.3: the dual of a wheel graph is a wheel.
51. Show that the graphs in Fig. 15.5 are isomorphic, but their duals are not isomorphic.
52. Exercise 15.6.i: Give an example to show that, if  $G$  is a disconnected plane graph, then  $G^{**}$  is not isomorphic to  $G$ .

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53. Prove Theorem 16.3 (Konig's Infinity Lemma).
54. Exercise 16.1: Various examples of infinite graphs. (see the book)
55. Exercise 16.2: Give an example to show that Konig's Infinity Lemma is false if we omit the condition that the infinite graph is locally finite.