

## Final Exam Review

The following sections will not be included in the final exam: 8, 11, 14, 15, 16, 18, 19, 26 (same can be said, of course, about sections which were not covered in the course, like: 1, 4, 24, 30-33).

1. Define the following terms:

- (a) (general) graph, simple graph
- (b) connected graph
- (c) cutset, bridge; edge connectivity  $\lambda(G)$
- (d) separating set, cut-vertex; connectivity  $\kappa(G)$
- (e) Eulerian trail; Eulerian graph, semi-Eulerian graph
- (f) Hamiltonian cycle; Hamiltonian graph, semi-Hamiltonian graph
- (g) forest, tree; spanning tree, spanning forest
- (h) planar graph
- (i) (valid) vertex coloring, chromatic number
- (j) (valid) edge coloring, chromatic index
- (k) chromatic polynomial
- (l) digraph
- (m) strongly connected (digraph)
- (n) orientable graph
- (o) tournament
- (p) matching, perfect matching, vertex cover
- (q) latin rectangle
- (r) network, capacity, flow, value of a flow, maximum flow, minimum cut

2. State the following theorems:

- (a) Euler's Theorem (for Eulerian graphs)
- (b) Ore's Theorem
- (c) Dirac's Theorem

- (d) Equivalent Definitions of a Tree
  - (e) Cayley's Theorem
  - (f) Kuratowski's Theorem
  - (g) Euler's Formula (for planar graphs)
  - (h) Brooks' Theorem
  - (i) Four Color Theorem
  - (j) Vizing's Theorem
  - (k) Konig's Theorem (on edge-colorings)
  - (l) Hall's Theorem
  - (m) Tutte's Theorem
  - (n) Konig-Egervary Theorem
  - (o) Menger's Theorem
  - (p) Max Flow Min Cut Theorem
3. Draw all non-isomorphic unlabelled graphs on 3 or 4 vertices.
  4. Given the degree sequence  $(3, 3, 5, 5, 5, 5)$ , does there exist a *simple* graph with these degrees?
  5. Draw, say how many vertices and edges are there and note if they are regular or bipartite, for the following graphs:  $K_n$ ,  $K_{n,m}$ ,  $W_n$ ,  $P_n$ ,  $C_n$ , the Petersen graph, the tetrahedron, the octahedron, the cube.
  6. Exercise 3.8:
    - (a) Prove that if  $G$  is self-complementary (isomorphic to its complement), then the number of vertices is of the form  $4k$  or  $4k + 1$ , where  $k$  is an integer.
    - (b) Find all self-complementary graphs on 4 and 5 vertices.
  7. If  $G$  is a bipartite graph, then any cycle has even length.
  8. Exercise 5.1: In the Petersen graph, find cycles of lengths 5, 6, 8 and 9; cutsets with 3, 4 and 5 edges.
  9. Find  $\kappa(G)$  and  $\lambda(G)$  for some graphs (e.g. Exercise 5.5)

10. Draw a graph  $G$  with minimum degree  $d$  such that

$$\kappa(G) < \lambda(G) < d$$

11. Exercise 5.9.
12. Exercise 5.10.
13. Show that if  $G$  is an Eulerian graph, then all degrees are even.
14. Use Euler's Theorem to show that a connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.
15. Given a graph, say if it is Eulerian or semi-Eulerian (e.g. Exercises 6.1 and 6.3).
16. Show that Ore's Theorem implies Dirac's Theorem
17. Given a graph, say if it is Hamiltonian or semi-Hamiltonian (e.g. Exercises 7.1 and 7.3)
18. Use Ore's Theorem to show that, if  $G$  is a graph with  $n$  vertices and  $(n-1)(n-2)/2 + 2$  edges, then  $G$  is Hamiltonian.
19. Prove Theorem 9.1.
20. From Theorem 9.1 we have that any tree on  $n$  vertices has exactly  $n-1$  edges.
- (a) Prove that any forest with  $n$  vertices and  $k$  connected components has exactly  $n-k$  edges.
  - (b) Prove that any graph with  $n$  vertices and  $k$  connected components has at least  $n-k$  edges.
21. Exercise 9.2: Show, by drawing, that there are (up to isomorphism) six trees on 6 vertices.
22. Exercise 9.3:
- (a) Prove that every tree is a bipartite graph.
  - (b) Which trees are complete bipartite graphs?

23. Exercise 9.4: Draw all the spanning trees in the graph of Figure 9.5
24. Exercise 10.4: How many spanning trees has the graph  $K_{2,n}$ ?
25. Show that Cayley's Theorem holds for  $n=4$  (See the beginning of Section 10).
26. Find the number of trees on  $n$  vertices in which a given vertex  $v$  has degree:
  - (a)  $\deg(v) = 1$ .
  - (b)  $\deg(v) = n - 2$ .
  - (c)  $\deg(v) = n - 3$ .
27. Theorem 12.1:  $K_5$  and  $K_{3,3}$  are not planar.
28. Given some graph, show whether it is planar or not. (e.g. Exercise 12.2).
29. Which complete graphs and complete bipartite graphs are planar?
30. Exercise 12.6: Prove that the Petersen graph is non-planar.
31. Prove Euler's Formula and use it to show that  $K_5$  and  $K_{3,3}$  are non-planar.
32.
  - (a) If  $G$  is a connected simple planar graphs with no triangles, then  $m \leq 2n - 4$ . (i.e. Corrolary 13.4.ii).
  - (b) Same question as in part (a), but now  $G$  contains no cycles of length less than 5.
  - (c) Use part (b) to show that the Petersen graph is non-planar.
33. Show that every simple planar graph contains a vertex of degree at most 5. (i.e. Theorem 13.6).
34. Given some plane drawing of a graph  $G$ , redraw the graph so that a given face becomes the infinite face. (e.g. Exercise 13.2).
35. Show that if  $G$  has at least 11 vertices, then  $G$  and  $\bar{G}$  cannot both be planar.

36. Prove theorem 17.1.
37. Prove theorem 17.3.
38. Prove the Five Color Theorem.
39. Find the chromatic number of some given graph. (e.g. Exercise 17.2)
40. Exercise 17.8.
41. Show that  $\chi'(K_9) = 9$  and that  $\chi'(K_{10}) = 9$ .
42. Show that  $\chi'(K_{6,9}) = 9$ . (note: there are 2 ways to prove this: by using Konig's theorem, or by exhibiting an explicit coloring)
43. Find the chromatic index of some given graphs (e.g. Exercise 20.2)
44. Show that if  $G$  is a cubic Hamiltonian graph, then it has chromatic index 3.
45. Prove that the Petersen graph has chromatic index 4 and deduce that it is not a Hamiltonian graph.
46. Exercise 20.10.
47. Show that if  $G$  is a simple graph and  $e \in E(G)$ , then  $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$
48. Find the chromatic polynomials for: any path graph, any tree, any complete graph,  $C_4$ ,  $C_5$ , any  $K_{2,n}$ .
49. Let  $G$  be a connected graph. Show that if each edge is contained in a cycle, then  $G$  is orientable.
50. Exercise 22.1: Given some digraphs argue whether or not they are isomorphic.
51. Show that every Hamiltonian graph is orientable.
52. Show, by finding an orientation, that the Petersen graph,  $K_n$  and  $K_{r,s}$  are orientable.
53. Show that every tournament is at least semi-Hamiltonian.

54. Show that every vertex in any tournament is included in a cycle of length exactly 3.
55. Exercise 23.6 on *irreducible* tournaments.
56. Exercise 23.7 on *transitive* tournaments.
57. Given some graph, argue whether or not Hall's condition holds and then find a maximum matching.
58. (a) Show that if  $G$  is a  $r$ -regular bipartite graph, then  $G$  has a complete matching.  
(b) The matching from (a) is actually a perfect matching. Why?  
(c) Deduce from (a) that the chromatic index of  $G$  is  $r$ .
59. Recast the problem of constructing Latin squares as a graph coloring problem.
60. (a) Give an example of a  $5 \times 8$  latin rectangle and a  $6 \times 6$  latin square.  
(b) Find a way to complete some latin rectangle to a latin square (e.g. Exercise 27.2)
61. Show that in a bipartite graph the size of any vertex cover is larger than the size of any matching. (i.e. the easy implication in Konig-Egervary)
62. Show that if  $G$  has a perfect matching, then,  $\forall S \subseteq V(G) : |S| \geq q(G - S)$
63. Petersen's Theorem: Show that any bridgeless cubic graph has a perfect matching.
64. Show that the Konig-Egervary Theorem (as stated in the handouts) implies Hall's Theorem
65. Show that Menger's theorem implies Hall's theorem.
66. Verify that Menger's theorem holds for some given graph.
67. Show that Menger's theorem implies the MFMC theorem.
68. Show that the MFMC theorem implies Hall's theorem.
69. Verify the MFMC theorem for some given network.