Handout 1 – Solution to the mixing problem

Problem. A bathroom sink contains 10 liters of water. Someone pours a mixture of sugar and water into the sink. The concentration of sugar in this mixture is 3 grams per liter. The mixture is poured at a rate of 1/2 liter per minute. At the same time, fluid is drained from the sink at a rate of 1/2 liter per minute. What is the concentration of sugar in the sink water after 7 minutes if initially it is 8 grams per liter?

Solution. Let $y(t)$ be the amount of sugar (measured in grams) in the sink at time t. The rate at which this amount changes is

$$\begin{aligned} y'(t) &= \frac{3 \text{ grams}}{\text{liter}} \cdot \frac{1/2 \text{ liter}}{\text{minute}} - \frac{y(t)\text{grams}}{10 \text{ liters}} \cdot \frac{1/2 \text{ liter}}{\text{minute}} \\ &= \frac{3}{2} - \frac{y(t)}{2} \\ &= \frac{30 - y(t)}{20}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dy}{30-y} &= \frac{dt}{20} \\ \int \frac{dy}{30-y} &= \int \frac{dt}{20} \\ -\log|30-y| &= \frac{t}{20} + C_1 \\ \log|30-y| &= -\frac{t}{20} + C_2 \\ |30-y| &= C_3 e^{-t/20} \\ y - 30 &= C_4 e^{-t/20} \\ y(t) &= C_4 e^{-t/20} + 30. \end{aligned}$$

From the initial condition $y(0) = 80$ grams, $C_4 = 50$ and after 7 minutes the amount of water is

$$y(7) = 50e^{-7/20} + 30.$$

Thus, the concentration of water after seven minutes is

$$5e^{-7/20} + 3 \text{ grams per minute.}$$

Notice that as $t \longrightarrow \infty$, $y(t) \longrightarrow 30$, so the concentration of sugar in the sink tends to 3 grams per liter, which is precisely the concentration of sugar in the mixture that is being poured into the sink. This makes sense: regardless of the initial concentration of sugar in the sink, after a long enough time the concentration of sugar in the sink should be very similar to that of the added mixture.