Fact Sheet

This fact sheet is for use throughout the course. During exams and quizzes (including the oral quiz) – when no notes or books will be allowed – a freshly printed copy of this fact sheet will be provided by the instructor. Students are encouraged to use their personal copy freely when doing any work related to the course, including homework, workshops, projects, or any other activity, be it inside or outside of the classroom. Students must try not to rely on notes written by hand on their personal copy of the fact sheet since these annotations will not be available during exams and quizzes.

1. Euler's method for a given step size Δt.
a. For a single differential equation $y' = f(t, y)$.

$$\begin{aligned} t_{k+1} &= t_k + \Delta t && \text{for all } k \\ y_{k+1} &= y_k + \Delta t f(t_k, y_k) && \text{for all } k. \end{aligned}$$

b. For a system $Y' = F(t, Y)$ of differential equations:

$$\begin{aligned} t_{k+1} &= t_k + \Delta t && \text{for all } k \\ Y_{k+1} &= Y_k + \Delta t F(t_k, Y_k) && \text{for all } k. \end{aligned}$$

2. Linearization theorem for an autonomous differential equation $y' = f(y)$ with equilibrium point y_0.

$$\begin{aligned} f'(y_0) > 0 &\implies y_0 \text{ is a source} \\ f'(y_0) < 0 &\implies y_0 \text{ is a sink .} \end{aligned}$$

3. Possible bifurcation values for a family $y' = f_\mu(y)$ of autonomous differential equations.

$$\begin{cases} f_\mu(y) = 0 \\ \frac{\partial}{\partial y} f_\mu(y) = 0. \end{cases}$$

4. Integrating factor for the linear differential equation $y' + g(t)y = b(t)$.

$$u(t) = e^{\int g(t)dt}.$$

5. The quadratic formula. Roots of $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

6. The inverse of a 2×2 matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{if } ad - bc \neq 0.$$

7. Characteristic polynomial of a 2×2 matrix A.

$$f(\lambda) = \det(A - \lambda I) = \lambda^2 - T\lambda + D.$$

8. Solution to initial value problem $dY/dt = AY$, $Y(0) = Y_0$.

$$Y(t) = e^{At} Y_0.$$

9. Decomposition $A = S\Lambda S^{-1}$ with Λ upper triangular.

Two real eigenvalues: $A = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} S^{-1}$, where
v_1 is an eigenvector with eigenvalue λ_1, and
v_2 is an eigenvector with eigenvalue λ_2.

Two complex eigenvalues: $A = \begin{pmatrix} v & \bar{v} \end{pmatrix} \begin{pmatrix} \lambda & \\ & \bar{\lambda} \end{pmatrix} S^{-1}$, where
v is an eigenvector with eigenvalue λ.

One repeated eigenvalue: $A = \begin{pmatrix} v & w \end{pmatrix} \begin{pmatrix} \lambda & c \\ & \lambda \end{pmatrix} S^{-1}$,
v is an eigenvector,
w is linearly independent of v, and
$Aw - \lambda w = cv$.

10. The exponential of Λ.

Two real eigenvalues: $e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix}$.

Two complex eigenvalues: $e^{\Lambda t} = \begin{pmatrix} e^{\lambda t} & \\ & e^{\bar{\lambda} t} \end{pmatrix}$.

One repeated eigenvalue: $e^{\Lambda t} = \begin{pmatrix} e^{\lambda t} & cte^{\lambda t} \\ & e^{\lambda t} \end{pmatrix}$.

11. The exponential of A from its decomposition.

$$e^{At} = Se^{\Lambda t} S^{-1}.$$

12. General solution to $dY/dt = AY$ via matrix exponentials.

$$Y(t) = Se^{\Lambda t} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \qquad k_1, k_2 \text{ arbitrary constants.}$$

13. Form of the general solution to $dY/dt = AY$ in the case where the 2×2 coefficient matrix has one repeated real eigenvalue.

$$Y(t) = e^{\lambda t} u_0 + te^{\lambda t} u_1,$$

where u_0 is an arbitrary vector and $u_1 := Au_0 - \lambda u_0$ is an eigenvector.

14. Equation of the repeated root parabola.

$$D = \frac{T^2}{4}.$$

15. Polar form of a complex number z.

$$z = |z|e^{i\arg(z)}.$$

16. Modulus and argument of the inverse of a non-zero complex number z.

$$|z^{-1}| = \frac{1}{|z|}$$

$$\arg(z^{-1}) = -\arg(z).$$

17. Frequency of beats and of rapid oscillations for a forced undamped harmonic oscillator with natural frequency $\frac{\sqrt{q}}{2\pi}$ and forcing frequency $\frac{\omega}{2\pi}$.

Frequency of beats: $\frac{1}{4\pi}|\omega - \sqrt{q}|$.

Frequency of rapid oscillations: $\frac{1}{4\pi}(\omega + \sqrt{q})$.

18. Jacobian matrix for the non-lynear system of differential equations $dx/dt = f(x,y), dy/dt = g(x,y)$.

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

19. Method for the classification of a fixed point x_0 of the discrete dynamical system $x_{n+1} = F(x_n)$.

$$\begin{array}{lcl} |F'(x_0)| > 1 & \Longrightarrow & x_0 \text{ is repelling} \\ |F'(x_0)| < 1 & \Longrightarrow & x_0 \text{ is attracting .} \end{array}$$