1. (25 points) Let us consider the geometrical sequence with the recursive definition \( a_{k+1} = qa_k \).

1) Give the formula for \( a_n \) through \( a_1 \) and \( q \) and prove it by the method of mathematical induction.

2) By the same method to prove the formula for the sum of \( n \) terms:

\[
S_n = a_1 + \cdots + a_n + \cdots = \frac{a_1(q^n - 1)}{q - 1}.
\]

1. \( a_n = a_1 q^{n-1} \).

   (i) \( a_1 = a_1 \).

   (ii) \( a_k = a_1 q^{k-1} \Rightarrow a_{k+1} = a_1 q^k \) \\

   \( a_k = a_1 q^{k-1} \Rightarrow a_{k+1} = a_1 q^k \) \hspace{1cm} \text{True}

2. \( S_n = a_1 + \cdots + a_n + \cdots \)

   (i) \( S_1 = a_1 \)

   (ii) \( S_n = \frac{a_1(q^n - 1)}{q - 1} \Rightarrow S_{n+1} = \frac{a_1(q^{n+1} - 1)}{q - 1} \)

   \( S_{n+1} = \frac{a_1(q^{n+1} - 1)}{q - 1} + a_1 q^n \) \hspace{1cm} \text{True}
2. (25 points) Prove that

\[ S_n = 1 + 3^3 + 5^3 \ldots + (2n-1)^3 = n^2(2n^2 - 1). \]

(i) \( n = 1 \Rightarrow 1^3 = 1 \); True

(ii) \( S_k = k^2 (2k^2 - 1) \Rightarrow S_{k+1} = \)

\[ = (k+1)^2 (2(k+1)^2 - 1) \]

\[ = (k^2 + 2k + 1)(2k^2 + 4k + 2) = 2k^4 + 8k^3 + 11k^2 + 6k + 1 \]

\[ S_k \Rightarrow S_{k+1} = k^2 (2k^2 - 1) + 6(2k+1)^3 + (2k+1)^2 = \]

\[ = 2k^4 - k^2 + 8k^3 + 11k^2 + 6k + 1 = \]

\[ = 2k^4 + 8k^3 + 11k^2 + 6k + 1. \]
3. (25 points) Prove that:

$$24|(n^2 - 1)n(n+2)$$

by two methods:
- directly, using properties of divisibility and
- by the method of mathematical induction.

1) **Direct.** \((n^2 - 1)n(n+2) = (n-1)n(n+1)(n+2)\).

The product of 4 consequent natural numbers is:
- at least 1 multiple 3,
- at least 2 consequent even (one multiple 4).

2) **Mathematical Induction.**

(i) \(n = 5, \ 24 | 10\)  True

(ii) \(24 | (k^2 - 1)k(k+2) \Rightarrow 24 | k(k+2)(k+2)(k+3)
\[= (k-1)k(k+1)(k+2)\]

(iii) \(6-\alpha = k(k+1)(k+2) \left[ k+3 - k+1 \right] = 4 k(k+1)(k+2)\)

Prove that \(6 | \alpha = k(k+1)(k+2)\)
- at least one factor multiple 3
- \(\alpha\) is even

\(6! 6-\alpha \ 3 \ 24 | \alpha \Rightarrow 24 | 6\)
4. (25 points) Show that any exact amount of postage that is an integer number of cents greater than 37 cents can be formed using just 8-cent and 5-cent stamps. (Prove by induction).

Basic Relations:

\[ 8 \cdot 2 - 5 \cdot 3 = 1 \]
\[ 5 \cdot 5 - 8 \cdot 3 = 1 \]

\[ k = 38 \quad n = 8a + 5b \quad a \geq 0, b \geq 0 \]

(i) \( n = 38 \), \( 38 = 8 + 5 \cdot 6 \), \( a = 1, b = 6 \)

(ii) \( k = 8a + 5b \) \( n \geq 38 \) \( (\exists \alpha)(\exists \beta) \)

1) \( b \geq 3 \) \( \Rightarrow \) \( \beta = b - 3 \) \( \alpha = a + 2 \)

2) \( b \leq 2 \) \( \Rightarrow \) \( 8a \geq 2 \cdot 8 \) \( \Rightarrow \) \( a \geq 1 \) \( \alpha = a - 3 \) \( \beta = b + 5 \)
5. (25 points) Prove that the formula

$$(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y);$$

is tautology but

$$(\exists x)(\forall y)P(x, y) \iff (\forall y)(\exists x)P(x, y)$$

is not.

1) \( A \Rightarrow B \). Let \((\forall y)P(x_0, y)\) is True.
\( A \Rightarrow \# \) for some \( x_0 \).

Then take \( a \) at \( B \) \( \#(y) \equiv x_0 \) and \( B \) True.

2) Let \( P(x, y) = x \cdot x + y = 0 \).

\( B \) is True: \( x(y) = -y \).

\( A \) is False.
6. (25 points) 1) Write down as a predicative formula with quantifiers only on the set of all real numbers the statement:
   For each nonnegative real number x exists and unique such y that $y^2 = x$.

2) Write down in the convenient form negation of this statement and prove it (in predicative form).

1) $\forall x \ (x \geq 0 \Rightarrow \exists! y \ y = \sqrt{x})$

2) $\exists x \ (x \geq 0 \land [\exists y \ y^2 = x \lor \exists u \exists v \ (u^2 + v^2 = x)])$

2) is True: Take any $x \geq 0$; let $x = 4$, $u = 2$, $v = 0$.

Then 2nd term of the disjunction is true:

$(2)^2 = 4 \land (-2)^2 = 4 \land 2 = -2$. 
7. (25 points) 1) State Peano's axioms of Arithmetic.
   2) Using directly by Peano axioms give the definition of the function on natural numbers \( f(n) = 2n \).

1) \[\begin{align*}
1 \in \mathbb{N} & \quad \text{special element} \\
\sigma(x, y) & \quad y = \text{successor of } x.
\end{align*}\]

(i) \( \forall x \in \mathbb{N} \)
(ii) \( (\exists x) (\exists ! y) \sigma(x, y) \)
(iii) \( (\forall x) (\forall y) (\forall z) (\sigma(x, z) \lor \sigma(y, z)) \Rightarrow x = y \)
(iv) \( (\forall x) (\exists y) \sigma(x, y) \Rightarrow y \neq 1 \)
(v) \( \{ P(1) \land (\forall x) (\forall y) (P(x) \land \sigma(x, y)) \Rightarrow P(y) \} \Rightarrow (\forall x) P(x) \)

2) \( f(x) \in \mathbb{N} \)
(i) \( \sigma(1, x) \equiv f(1) = x \)
(ii) \( f(k) = (x \land \sigma(x, y) \land \sigma(y, z)) \equiv f(k) \land \sigma(k, k+1) \)
\( \Rightarrow f(k+1) = 2 \)
\( y = k+1 \equiv \sigma(k, y) \).
8. (25 points) 1) Define in the predicative form that \( \lim_{n \to \infty} a_n = L \).
2) Give the convenient negation of this definition.
3) Define that a sequence \( \{ a_n \} \) diverges (has no limit).
4) Find \( \lim_{n \to \infty} 2/n \) and prove it in the language of \( \varepsilon, N \).

1) \( \forall \varepsilon \exists N \forall n (n > N \Rightarrow |L - a_n| < \varepsilon) \)

2) \( \exists \varepsilon \forall \varepsilon > 0 \forall n \exists N (n > N \land |L - a_n| > \varepsilon) \)

3) \( \forall L \exists \varepsilon \)

4) \( \lim_{n \to \infty} \frac{2}{n} = 0 \).

Proof. \( \frac{2}{n} < \varepsilon \iff \frac{2}{\varepsilon} < n \). Let \( N(\varepsilon) = \lceil \frac{2}{\varepsilon} \rceil \).

(\forall) \( n > N(\varepsilon) \) \( \Rightarrow \) \( \frac{2}{n} = 10 - \frac{2}{n} < 0 \).