Lecture 13.6

1.3

\[ (\exists k \Rightarrow (\exists a)) \text{ we can omit } (\exists a). \]

\[ (\exists a) (m = 2a + 1 \land m = 4k + 1) \Rightarrow (\exists j) m + 2 = 4j - 1 \]

You can remove it since \( \exists k \Rightarrow \exists j. \)

\[ m \text{ is only free subject variable; other ones connected by quantifiers}. \]

Proof. Remar:\( k \geq 2k. \)

\[ m = 4j - 2 \quad \Rightarrow \quad m = 4k + 1 \Rightarrow m + 2 = 4k + 3 = \]

\[ = 4(k + 1) - 1 \Rightarrow m + 2 = 4j - 1, \quad j = k + 1 \Leftrightarrow Q. \]

\( (\exists i) (\forall k) m = 2k + 1 \Rightarrow (\exists k) m^2 = 8k + 1 \quad Q \]

\[ P \Rightarrow (\exists i) m = 4i^2 + 4i + 1 = 4i(i + 1) + 1 \Rightarrow \]

\[ (\exists j) m = 8j + 1 \Leftrightarrow Q. \]

Intermediate Theorem: \( (\forall i) 2i(i + 1) \Leftrightarrow (\forall i)(\exists j) \]

\[ i(i + 1) = 8j \]
Lemma. Odd numbers can have in division on 4 remainder 1 or 3.

Corollary. If \( n = 2a + 1 \) then

\[
(36) \quad n = 4b + 1 \vee n = 4b - 1.
\]

Proof. \( 2 \Rightarrow 0 \). By contradiction

\[
(32) \quad n = 4a + 1 \quad n = 4b + 1 \Rightarrow
\]

\[
mn = 16ab + 8(a + b) + 1 = 4c + 1 \Rightarrow \not{p}.
\]

\[
3. \quad (\forall m) \quad n = 2k \wedge n > 2 \quad \Rightarrow \quad (p_1) (p_2)
\]

\[
h = p_2 + p_2 \Rightarrow (\forall m) (36) \quad m = 2k + 1 \wedge m > 5 \Rightarrow
\]

\[
(p_3) (p_4) (p_5) \quad m = p_2 + p_3 + p_4.
\]

Proof. \( m \mid m = m - 3 \Rightarrow \) \( n \) even \( n > 2 \Rightarrow \)

\[
m = 3 + p_4 + p_2.
\]
1) \( (\forall x) (\forall y) \ (x > 1 \land y > 0 \Rightarrow y^x > x^y) \) \[12\]

\[ y = \frac{1}{x} \quad \frac{1}{x} < 1 \quad x > 1 \quad \Rightarrow \quad \left( \frac{1}{x} \right)^x < 1 < x. \]

2) \( (\exists y) (\forall x) \ y > 0 \land x^y > x \forall y \leq x \]

3) \( (\forall x) \ x > 0 \Rightarrow x^2 - x > 0. \quad \boxed{\text{F}} \)

4) \( (\exists x) \ x > 0 \land x^2 < x \quad x = \frac{1}{2} \)

But \( (\forall x) \ x > 1 \Rightarrow x^2 - x > 0 \) \( \boxed{\text{T}} \)

Remember! \( \sim (P \Rightarrow Q) \Leftrightarrow P \land \sim Q \)

5) \( (\forall x) \ x > 0 \Rightarrow 2^x > x + 1 \quad \boxed{\text{F}} \)

\[ x = \frac{1}{2} \quad \sqrt{2} < \frac{3}{2} \]

But \( T \) if \( x > 1 \)

\[ \#5 \ a. \ [\text{LN}] \quad (\forall y) \ y \mid x \Rightarrow \exists y, y \in \mathbb{N} \ (1 \leq y \leq \sqrt{x}) \]

\[ [\text{LN}] \quad x > 1 \land \sim (\exists y) \ y \mid x \land 1 \leq y \leq \sqrt{x} \Rightarrow \]

\[ x > 1 \land \forall y \ y \mid x \Rightarrow (y = 1 \lor y > \sqrt{x}) \quad \boxed{(1)} \]

Usual definition of prime numbers:

\[ x > 1 \land \forall y \ y \mid x \Rightarrow y = 1 \lor y = x. \quad \text{We need to prove equality}. \]

Idea: if \( y \)-divisor and \( y > \sqrt{x} \), then \( \frac{x}{y} < \sqrt{x} \).
Proof. Contradiction. Let (1) is true and
let \( y \lor x \), \( y \neq 1, y \neq x \).

Take \( z = \frac{x}{y} \). Then \( z \neq 1 \land z < \sqrt{x} \).
Contradiction with (1)!

b) \( \exists P \{ \text{prime numbers} \}

(\forall p)(p \in P \land p \neq 3 \implies 3 \mid p^2 + 2)

Lemma 2. \( \forall n \ 3 \mid n \implies \exists h \ n = 3k + 1 \lor n = 3k - 1 \).

Proof. If remainder is 2 \( \implies n = 3k - 1 \).

L.H. \( \forall n \ 3 \mid n \implies \exists k \ n^2 = 3 \mid k + 1 \).

If \( 3k \ n = 3k + 1 \lor n = 3k - 1 \implies n^2 = 9k \pm 6k + 1 \).

\( \implies 3 \mid n^2 + 1 \). (or \( 3 \mid n^2 - 1 \)).

Remark. It's not essential that \( p \) is prime; only \( 3 \mid n \)
6 d) \((\exists M)\ (M \in \mathbb{N} \land (\forall n) \frac{1}{n} < M)\)

Proof. \(M = 1\).

e) \(\forall n (\exists m) n > m \Rightarrow \forall m (\exists n) n \leq m\).

Proof. \(m = n\).

Extra Problem.

1. Each sage starts: If my cap is white
   2 friends of
   - one white and one black caps.

2. What do they think:
   If my cap is white than one sage see
   2 white caps and he immediately will say: My cap is black.
   3. After a moment: E nobody did it!

So sages understand that nobody see
2 white caps, after 2 E they understand that
no white caps at all.

They are equally smart since they answered simultaneously. So for all period E is the
best.
1. Extra problem

2. Sec. 1.6

4th, 11, 5 (a, b)

General principles

1. In definitions of predicates \( P(x_1, \ldots, x_n) \), we write formulas with free variables (subjects) \( x_1, \ldots, x_n \). All other variables are connected.

2. Theorems are tautologies on the objective fields; all variables are connected.

3. At both definitions and theorems only already defined predicates participated. They must be at the convenient form (negations only elementary predicates).

Remark. \( a = bq + r \), \( 0 \leq r < b \).

Sometimes convenient to take negative remainder \((-b + r) = -r\)
4. Some relations between quantifiers

1) \((\forall x)(\forall y)P(x,y) \Leftrightarrow (\forall y)(\forall x)P(x,y)\)
2) \((\exists x)(\exists y)P(x,y) \Leftrightarrow (\exists y)(\exists x)P(x,y)\)
3) \((\exists x)(\forall y)P(x,y) \Rightarrow (\forall y)(\exists x)P(x,y)\)

Inverse can be not True!
So the order of quantifiers is essential

\[(\forall x)(\exists y)(P(x) \land P(y)) \Leftrightarrow (\forall y)(\exists x)P(x)\]

4) \((\forall x)\,(P(x) \land Q(x)) \Leftrightarrow (\forall y)P(y) \land (\forall z)Q(z)\)

5) \((\forall x)\,(P(x) \lor Q(x)) \Leftrightarrow (\forall y)\,(P(y) \lor (\forall z)Q(z))\)

Why?

Only \(\Leftrightarrow\) True

6) \((\exists x)\,(P(x) \lor Q(x)) \Leftrightarrow (\exists y)\,(P(y) \lor (\exists z)Q(z))\)

7) \((\exists x)\,(P(x) \land Q(x)) \Leftrightarrow (\exists y)\,(P(y) \land (\exists z)Q(z))\)

Only \(\Rightarrow\) True

In these formulas, \(P\) is a predicative variable.
When we say "it is true" we mean that it's Tautology (for any \(P\)).
"not True" = neither Tautology nor Contradiction.
We need to build a counterexample of \(P\).
5. It's possible to put all quantifiers in the front of formula in correct order. When we proved Theorem we move along quantifiers from left to write tight choosing for each \((\exists x)\ x\) as function of all already chosen variables (for quantifies on the left).

Example:

\((\forall x)\ (\forall y)\ (\exists z)\ (\forall w)\ \ldots\)

\[z = f(x, y)\]
\[\forall x \forall y : g(x, y) \equiv f(x, y, w)\]