Lecture 13.

Properties of quantifiers.

1. Unique existential quantifier:
   \[(\exists! x) P(x) \] exists a unique \( x \) s.t. \( P(x) \) is True.
   \[(\exists! x) P(x) \iff (\exists x) P(x) \land (\forall y)(\forall z)(P(y) \land P(z) \Rightarrow y = z).\]
   Ex. \((\forall x)(\exists! y) y^2 = x \quad (\mathbb{R}^+)\)
   \((\forall y)(\exists! x)(y = 2^x) \quad [x: \mathbb{R}; y: \mathbb{R}^+].\)

2. Using brackets always show areas of actions of quantifiers.
   Ex. \((\forall x)(\exists y)(P(x, y) \lor (\exists y)(Q(y))).\)
   \((\forall x)(x = 0 \lor (\exists y) y^2 = \frac{1}{x}) \quad (\mathbb{R}).\)
   \((\forall x)(\exists y) A(y) \lor (\forall y) A(y))\)
Let the subject field be finite.

Then

\((\forall x) P(x) \iff P(x_1) \land \cdots \land P(x_n)\)

\((\exists x) P(x) \iff P(x_1) \lor \cdots \lor P(x_n)\).

4. Similar quantifiers can be interchanged:

\((\forall x)(\forall y) P(x, y) \iff (\forall y)(\forall x) P(x, y)\)

\((\exists x)(\exists y) P(x, y) \iff (\exists y)(\exists x) P(x, y)\)

But non-similar — not

\((\forall x)(\exists y) P(x, y) \iff (\exists y)(\forall x) P(x, y)\)
\( P(x, y) \): Student \( x \) takes course \( y \).

\( \forall x \exists y \ P(x, y) \): Each student takes some course.

\( \exists y \forall x \ P(x, y) \): Some student takes all courses.

Remark: \( \exists y \forall x \ P(x, y) \Rightarrow \forall x \exists y \ P(x, y) \) is true!

\( x = \mathbb{R}^+ \) \( \forall x \ P(x, y) \) is true

but \( \exists x \forall y \ P(x, y) \) is False.

\[ \sim (\forall x) P(x) \iff (\exists x) (\sim P(x)) \]

\[ \sim (\exists x) P(x) \iff (\forall x) (\sim P(x)) \]

Not all cats are gray \iff all cats are not gray.

Ex. Not all polynomials have real roots.

\[ \exists x \text{ There is a polynomial without real roots.} \]

It is similar to convenient formulas for propositional connectives.