Introduction to
Mathematical Reasoning

640: 300
Fall 2015
Lecture 1.

Propositions and Connectives.

Informal definition: A proposition is a sentence about which we can say that it is either true or false (true value).

Examples of sentences which are propositions:

- He is a student.
- What I said now is a lie.
- Hillary Clinton will be President of U.S.

Notations for propositions: Capital letters! A, B, C, ..., X, Y, ...
Exercise for kids.

One of kids - John, Joe and Lisa - broke a cup. Their mother asked them who did it. John said that Lisa did it. We don't know what Joe and Lisa answered. The mother found who was guilty and it turned out that only this child said the truth. Who did broke the cup?

[John said a lie? Joe did it]
Propositional algebra

Propositional operation or connective \( f(a, b) \) is a compound proposition from \( a, b \) such that the true value of \( f(a, b) \) is defined by true values of \( a \) and \( b \).

Examples.

**Conjunction** \( P \land Q \) is true iff \( P \) and \( Q \) are true.

**Disjunction** \( P \lor Q \) is false iff \( P \) and \( Q \) are false.

**Negation** \( \neg P \) is true iff \( P \) is false.

\[ \begin{align*}
P \land Q & : \quad \text{"P and Q"} \\
P \lor Q & : \quad \text{"P or Q"} \\
\neg P & : \quad \text{"not P"} \end{align*} \]
Attention: P \lor Q correspond "alternative or".
Non-alternative "either ... or ..."

a. Compound \( \land \) propositions don't care about a common sense but only about a possibility to give a true value.

Truth tables

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \land Q</th>
<th>P \lor Q</th>
<th>\sim P</th>
</tr>
</thead>
<tbody>
<tr>
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We can consider connectives for many propositions

\( f(a_1, \ldots, a_n) \).
Ex.
P - "The sun is shining outside"
Q - "A class is underway in the classroom"

Compound propositions (connective \( \land \))

\[ P \land Q \]
\[ P \land (\neg Q) \]
\[ (\neg P) \lor (\neg Q) \]

Different grammatical connectives correspond to the same propositional one:

Conjunction \( \land \) - and, but, while

Analogy:

Numerical function \( f(x_1, \ldots, x_n) \)
Variables - numbers

Propositional function (connective \( f(P_0, \ldots, P_n) \))
Propositional functions (connectives)

\[ f(P_1, \ldots, P_n) \text{ and } g(P_1, \ldots, P_n) \]

are called equivalent if they have identical truth tables.

Remarks: \( P \lor Q \) and \( R \lor S \) are not equivalent; variables must coincide.

Propositional formula (form) - a composition of connectives:

\[ F_x (P \land (\sim Q)) \land (\sim Q \lor \sim R) \]

Properties of connectives - equivalences of propositional formulas
Basic properties (laws) of $\land$, $\lor$, $\sim$.

1. Double negation $\sim(\sim P) \equiv P$.

2. Commutative laws
   \[ P \land Q \equiv Q \land P \]
   \[ P \lor Q \equiv Q \lor P. \]

3. Associative laws
   \[ P \land (Q \land R) \equiv (P \land Q) \land R \]
   \[ P \lor (Q \lor R) \equiv (P \lor Q) \lor R. \]

4. Distributive laws
   \[ P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R) \]
   \[ P \lor (Q \land R) \equiv (P \lor Q) \land (P \lor R). \]

5. De Morgan's laws
   \[ \sim(P \land Q) \equiv \sim P \lor \sim Q \]
   \[ \sim(P \lor Q) \equiv \sim P \land \sim Q. \]
Proofs.

The universal way to prove is to compare the truth table for the left and right parts.

Ex. 1. Distributive law 6) $P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P \land (Q \lor R)$</th>
<th>$(P \land Q) \lor (P \land R)$</th>
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8 lines (Why?)

The last columns coincide so these two connective formulas equivalent.
Often we can replace truth tables by equivalent narratives.

**Example:** $P \land Q \equiv Q \land P$ since both connectives are true if both $P, Q$ are true (independently of the order).

2) $P \lor (Q \lor R)$ and $(P \lor Q) \lor R$ are both false iff all $P, Q, R$ are false.

3) $\neg (P \land Q)$ is false iff $P \land Q$ is true, s.t. iff both $P$ and $Q$ are true.

Correspondingly, $(\neg P) \lor (\neg Q)$ is false iff both $\neg P$ and $\neg Q$ are false s.t. if both $P$ and $Q$ are true.

So we have $\neg (P \land Q) \equiv (\neg P) \lor (\neg Q)$. 
All laws 1) - 5) have a deep logical sense

1) Double negation is equivalent to statement

2) Commutative laws allow to change the order at conjunctions and disjunctions

3) Associative laws allow consider conjunctions (disjunctions) with many terms without parentheses:

   $P_1 \land P_2 \land \ldots \land P_n$

   $Q_1 \lor Q_2 \lor \ldots \lor Q_n$

4) 5) Distributive laws allow "to open brackets"; "distribute conjunction on terms of disjunction and

   opposite:

   compare with numbers!

   You can distribute multiplication on addition: $x(y+z) = xy + xz$ but not
opposite \( x + yz \neq (x + y)(y + z) \).

But at Propositional Algebra we have 2 distributive laws.

De Morgan laws

The great logical principle: compound
Never start a statement from the negation!

It's not a mistake but not informative!

Ex. It's not true that
John likes Mathematics and
Lisa likes Biology.

How to transform it in "convenient" form?

At the textbook: "useful"
John doesn't like Mathematics or Liz doesn't like Biology.

Pay attention and $\Rightarrow$ or.

It's exactly De Morgan law

$P$ - John likes Mathematics
$Q$ - Liz likes Biology

Our statements: $\sim (P \land Q)$
We transform at the "convenient" form

$(\sim P) \lor (\sim Q)$.

Principle: Push negations inside of statements.
