Fei Qi
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Grading breakup:

Homework: 100 pts
Presentation: 50 pts  Everyone must present at least once
Midterm x 2: 100 pts x 2. Graded pass/fail.
Final: 200 pts. Discount might be applied for low attendance,
sloppy HW makeup, etc.

Reference:

W. Rudin. Principle of Mathematical Analysis (classical)
V. Zorich. Mathematical Analysis (modern)
A. Mattuck. Introduction to Analysis (easy, wordy)
Polya - Szegö. Problems and Theorems in Mathematical Analysis.

What Analysis studies: Real-valued functions of real variable(s).
or in short, f: \mathbb{R} \to \mathbb{R}

A priori, one has to understand \mathbb{R}.
Geometrically, real numbers \leftrightarrow points on the number line.
Given a point \( x \) on the \( x \)-axis, a function specifies a point \( y \) on the \( y \)-axis for the point \( x \) by \( y = f(x) \).

As \( x \) ranges through \( x \)-axis, all \( (x, f(x)) \) on \( xy \)-plane yields the graph of \( f \).

This has been the picture you are familiar with in Computation-Based Calculus. But Arithmetic is going to be trickier.

History of numbers:

- \( \emptyset \) no two fingers are identical
  - abstraction of notion.
- \( \mathbb{Z}^+ \) positive integers
  - we can count the "number" of fingers, or eggs, or leaves, etc.
  - joining zero
- \( \mathbb{N} \) natural numbers
  - addition \( \checkmark \), multiplication \( \checkmark \)
  - subtraction \( x \), e.g. \( x \in \mathbb{N}, a+3=1 \)
- \( \mathbb{Z} \) integers
  - addition \( \checkmark \), multiplication \( \checkmark \), subtraction \( \checkmark \)
  - division \( x \), e.g. \( a \in \mathbb{Z}, a \cdot 3 = 1 \)
- \( \mathbb{Q} \) rational numbers. \((+, -, \times, 
\) completely defined.

Remark: Each extension above somehow "completes" an operation. More precisely, some operation is completely defined after some extension.
Question: does rational number represent all possible lengths of segments?

Example: Let $c$ be the length of a diagonal of the unit square.

Pythagorean Thm: $c^2 = 1^2 + 1^2 = 2$

Claim: $c$ is NOT a rational number.

Pf: Suppose otherwise that $c \in \mathbb{Q}$, then

$c \in \mathbb{Q} \Rightarrow \exists p, q \in \mathbb{Z},$ $p, q$ have no common factor, $c = \frac{p}{q}$

$c^2 = 2 \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2 \Rightarrow 2 \mid p^2$

$2$ is a prime, $2 \mid p^2 \Rightarrow 2 \mid p$

(In general, $k$ is prime, $k \mid ab \Rightarrow (k \mid a$ or $k \mid b)$)

Thus $\exists p_1 \in \mathbb{Z},$ $p = 2p_1$

Then $p^2 = 2q^2 \Rightarrow 4p_1^2 = 2q^2 \Rightarrow 2p_1^2 = q^2 \Rightarrow 2 \mid q^2 \Rightarrow 2 \mid q$.

The highlighted facts indicates $2$ is a common factor of $p$ and $q$, contradicting the choice of $p, q$.

Therefore $c \notin \mathbb{Q}$.

So to represent all possible lengths of segments, $\mathbb{Q}$ is insufficient.

We need another extension $\mathbb{Q} \to \mathbb{R}$. This will be the main topic in the next few classes.
Discuss: (1) Prove \( \sqrt{3} \notin \mathbb{Q} \)

(2) Modify the argument to show \( \sqrt{6} \notin \mathbb{Q} \)

(3) Let \( r \) be a number such that \( 2^r = 3 \), prove that \( r \notin \mathbb{Q} \).

Hint: With \( r = \frac{p}{q} \), rewrite \( 2^r = 3 \) so that both sides are integers, to find a contradiction.