Fei Qi

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Reference:

W. Rudin. Principle of Mathematical Analysis (classical)
V. Zorich. Mathematical Analysis (modern)
A. Mattuck. Introduction to Analysis (easy, wordy)
Polya- Szegö. Problems and Theorems in Mathematical Analysis.

Recall: A map \( f: A \rightarrow B \) assigns a unique element \( f(a) \in B \) for each \( a \in A \).

Example: \( A = \{ \text{sections in Math 311 class} \} \)
\( B = \{ \text{TA s in Math Dept.} \} \)

\((\text{Section } x) \xrightarrow{f} (\text{TA of the section } x)\) is a well-defined
map from \( A \rightarrow B \). e.g. \( f(\text{Section 2}) = (\text{TA Fei Qi}) \)

\((\text{TA y}) \xrightarrow{f} (\text{Section TA y is in charge of})\) is NOT a well-defined
map from \( B \rightarrow A \), b/c Fei Qi is in charge of two sections,
namely H1 and O2. Not unique!
When $A, B$ are sets of numbers, we call $f : A \to B$ a function. Historically, functions were introduced and studied prior to maps.

Example: $x \in \mathbb{R} \mapsto$ square root of $x$ is NOT a well-defined function.

$x \in \mathbb{R} \mapsto$ positive square root of $x$ is a well-defined function.

Think about why.

For a function $f : A \to B$,

- $A$ is referred as its domain, $B$ is referred as its codomain.
  * Properties may change if the domain or codomain changes for the same rule of assignment!

- For $X \subseteq A$ subset, the collection of all elements in $B$ that $A$ maps to is called the image of $X$, denoted $f(X)$.
  Formally: $f(X) = \{ y \in B : \exists x \in X \}$
  Equivalently: $y \in f(X) \iff \exists x \in X, y = f(x)$

- For $Y \subseteq B$ subset, the collection of all elements in $A$ that map into $Y$ is called the preimage of $Y$, denoted $f^{-1}(Y)$.
  Formally: $f^{-1}(Y) = \{ x \in A : f(x) \in Y \}$
  Equivalently: $x \in f^{-1}(Y) \iff f(x) \in Y$

Example: For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Find

- $f([0,\infty))$, $f([-9,16])$, $f([-1,4])$
- $f^{-1}([0,\infty))$, $f^{-1}([-9,16])$, $f^{-1}([-1,4])$

(Try it before looking at the answer next page.)
Ans: \( f(\mathbb{R}) = \mathbb{R}, \quad f([0, \infty)) = [0, \infty) \), \( f([0, 4]) = [0, 256] \), \( f([1, 4]) = [1, 16] \)
\( f'(\mathbb{R}) = \mathbb{R}, \quad f''([0, \infty)) = \mathbb{R}, \quad f''([0, 4]) = [0, 128], \quad f''([1, 4]) = [0, 4] \).

- \( f: A \rightarrow B \) is called injective if
  \[ \forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2. \]
Informally: If two numbers are mapped to the same number, then they are equal.
Equivalently: \( \forall x_1, x_2 \in A, \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \)
Informally, different numbers are mapped to different numbers.

- \( f: A \rightarrow B \) is called surjective if
  \[ \forall y \in B, \exists x \in A, \quad f(x) = y. \]
Informally: Every number in the codomain is the image of some number in the domain.
Equivalently: \( f(A) = B \).

- \( f \) is bijective if \( f \) is both injective and surjective.

Example: \( A, B \in \mathbb{R} \). Find if \( f: A \rightarrow B, f(x) = x^2 \) is injective, surjective or bijective.

1. \( A = \mathbb{R}, \quad B = \mathbb{R} \)
2. \( A = \mathbb{R}, \quad B = [0, \infty) \)
3. \( A = \mathbb{R}, \quad B = [-1, 4] \)
4. \( A = \mathbb{R}, \quad B = [1, 4] \)
5. \( A = [0, \infty), \quad B = \mathbb{R} \)
6. \( A = (-\infty, 0], \quad B = [0, \infty) \)
7. \( A = [-1, 4], \quad B = [1, 16] \)
8. \( A = [1, 4], \quad B = [1, 16] \)

I wish this example continues you that domain and codomain make a difference.
Try before looking at the next page.
Example: Let \( f: X \to A \) be a function, \( X \subseteq A \) be a subset.
Prove that \( X \subseteq f^{-1}(f(X)) \). Show by example that the inclusion can be proper, and prove that if \( f \) is injective, then \( X = f^{-1}(f(X)) \).

**Solution:**
1. \( x \in X \implies f(x) \in f(X) \)

 Denote \( Y = f(X) \subseteq B \), then \( f(x) \in Y \)

 By def. of \( f^{-1}(Y) \), we have \( x \in f^{-1}(Y) \), which is \( f^{-1}(f(X)) \).

 So we proved \( x \in X \implies x \in f^{-1}(f(X)) \).

 Thus \( X \subseteq f^{-1}(f(X)) \) \( \Box \)

2. Take \( f: R \to R \), \( f(x) = x^2 \). Let \( X = [-1,2] \). Compute \( f^{-1}(f(X)) \) to see.

 \( X \notin f^{-1}(f(X)) \) (Hint: Check the above examples for \( f^{-1}(f(X)) \))

3. We show \( x \in f^{-1}(f(X)) \implies x \in X \) when \( f \) is injective.

 \( x \in f^{-1}(f(X)) \implies f(x) \in f(X) \subseteq B \)

 Denote \( y = f(x) \in B \), then \( y \in f(X) \)

 By def. of \( f(X) \), \( \exists x' \in X \), \( f(x') = y \).

 In general, \( x' \) doesn't have to be \( x \). But now that \( f \) is injective,

 recall \( y = f(x) \), so \( f(x) = f(x') \implies x = x' \).

 Since \( x' \in X \), so \( x = x' \implies x \in X \).

 So we proved \( x \in f^{-1}(f(X)) \implies x \in X \)

 Thus \( f^{-1}(f(X)) \subseteq X \). \( \Box \).
Exercise: Let \( f: A \to B \) be a function, \( Y \subseteq B \) be a subset.

Prove that \( f(f^{-1}(Y)) \subseteq Y \). Show by example that the inclusion can be proper, and prove that if \( f \) is surjective, then \( f(f^{-1}(Y)) = Y \).

Exercise. \( \forall X \in A, X = f^{-1}(f(X)) \iff f \) is injective.

Well-ordering principle. Every nonempty subset of \( \mathbb{Z}_+ \) has a smallest member.

Principle of Math Ind.: \( P(1) \land (\forall n \in \mathbb{Z}_+, P(n) \Rightarrow P(n+1)) \Rightarrow (\forall n \in \mathbb{Z}_+, P(n)) \)

Proof: Assume \( \exists k \in \mathbb{Z}_+ \) \( \Rightarrow P(k) \).

Collect all such \( k \)'s to form a subset of \( \mathbb{Z}_+ \).

Name it \( S \). Well-ordering Principle \( \Rightarrow S \) has a smallest elt, say \( k_0 \).

i.e. \( \forall n < k_0, P(n) \) is true. In particular, \( P(k_0 - 1) \) is true.

From \( (\forall n \in \mathbb{Z}_+, P(n) \Rightarrow P(n+1)) \) with \( P(k_0 - 1) \) is true we know \( P(k_0) \) is true, i.e. \( k_0 \notin S \). Contradiction.

This proves \( \forall k \in \mathbb{Z}_+, P(k) \iff (\forall k \in \mathbb{Z}_+), P(k) \).

Example. \( (\forall n \in \mathbb{Z}_+), 1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} \). If \( x \neq 1 \).

\[ n = 1 \quad \text{LHS} = 1 + x, \quad \text{RHS} = \frac{1 - x^2}{1 - x} = 1 + x \]

Assume for \( n = k \), \( \text{LHS} = \text{RHS} \), i.e. \( 1 + x + \ldots + x^k = \frac{1 - x^{k+1}}{1 - x} \).

Want: \( n = k + 1 \), \( \text{LHS} = \text{RHS} \), i.e. \( \text{Want} \quad 1 + x + \ldots + x^k + x^{k+1} = \frac{1 - x^{k+2}}{1 - x} \)

From ind. hypo: \( \text{LHS} = \frac{1 - x^{k+1}}{1 - x} + x^{k+1} = \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x} = \frac{1 - x^{k+2}}{1 - x} = \text{RHS} \).

From principle of math induction, \( \text{LHS} = \text{RHS} \forall n \in \mathbb{Z}_+. \) \( \square \)
Exercise: Prove that \( n^3 + 5n \) is divisible by 6, \( \forall n \in \mathbb{Z}^+ \).
Recall: Cartesian Product $A \times B = \{(a, b): a \in A, b \in B\}$

Relation between $A, B$: any subset of $A \times B$.

Function: relation built on set of numbers, satisfying

$a \sim b_1, a \sim b_2 \Rightarrow b_1 = b_2 \in B$

$f: A \rightarrow B$. If $a$ is related to $b$, denote it by $f(a) = b$

the requirement means $a_1 = a_2 \in A \Rightarrow f(a_1) = f(a_2) \in B$

Given $f: A \rightarrow B$. Let $M \subseteq A$, $N \subseteq B$.

Image of $f$ on $M$ is the subset $\{y : f(a) \in B : a \in M\}$ of $B$

denoted $f(M)$

Preimage of $f$ on $N$ is the subset $\{a \in A : f(a) \in N\}$ of $A$.

Example: $f(x) = x^2$ gives a function $f : \mathbb{R} \rightarrow \mathbb{R}$

$f(\mathbb{R}) = \{y \geq 0\}$, $f([0, \infty)) = \{y \geq 0\}$, $f([2, 4]) = \{4 \leq x \leq 16\}$

$f^{-1}([0, \infty)) = \mathbb{R}$, $f^{-1}(\mathbb{R}) = \mathbb{R}$, $f^{-1}([9, 16]) = [-4, -3] \cup [3, 4]$

$f^{-1}([-1, 16]) = [-4, 4]$ $f^{-1}((\infty, 0)) = \emptyset$
In general: \( f: A \rightarrow B \text{ func.} \)
\[ \Rightarrow X \subseteq f^{-1}(f(x_1)). \text{ It might be a proper inclusion.} \]
\[ \text{e.g. } f: x \mapsto x^2. \quad X=[0,\infty). \quad f^{-1}(f([0,\infty)) = f^{-1}([0,\infty)) = \mathbb{R} \cup X \]

**Proof:** \( x \in X \), we should show \( x \in f^{-1}(f(x)) \)

Recall \( x \in f^{-1}(N) \iff f(x) \in N \)
\[ x \in f^{-1}(f(x)) \iff f(x) \in f(x), \text{ which is obvious if } x \in X. \text{ (backward)} \]

More formally: \( x \in X \Rightarrow f(x) \in f(x) \Rightarrow x \in f^{-1}(f(x)). \text{ (direct)} \)

Q.E.D.

**Exercise:** Prove that for \( Y \subseteq B \), \( f^2(f^{-2}(Y)) \subseteq Y \).

In some cases, \( X = f^{-1}(f(x)) \). Guess the condition for \( f \)?

Ans: \( f \) is injective.

**Proof:** It suffices to show \( f^{-1}(f(x)) \subseteq X \).
\[ x \in f^{-1}(f(x)) \Rightarrow f(x) \in f(x). \]

Recall: \( a \in f(x) \iff \exists y \in X, s.t. f(y) = a \).

(Replace \( a \mapsto f(a) \)). \( \Rightarrow \exists y \in X, f(y) = f(x) \)

Recall: \( f \) is injective iff \( f(x) = f(y) \) for some \( x, y \in A \Rightarrow x = y \).

\[ \text{e.g. } x \mapsto x^2 \text{ is injective as func. } [0, \infty) \rightarrow \mathbb{R} \]
\[ \text{not injective as func. } \mathbb{R} \rightarrow \mathbb{R}. \]

**Inj:** \( y = x \). Since \( y \in X \), \( \Rightarrow x \in X \).

**Exercise:** Prove that for \( Y \subseteq B \), \( f^2(f^{-2}(Y)) = Y \) if \( f \) is surjective.