Lecture #22:

- Qualitative Method for Systems of ODE
  \[ \begin{align*}
  x' &= F(x, y) \\
  y' &= G(x, y)
  \end{align*} \]
  RHS does not depend on \( t \)

  Autonomous System

  - With initial condition specified as:
    \[ \begin{align*}
    x(t_0) &= x_0 \\
    y(t_0) &= y_0
    \end{align*} \]

  \( \Rightarrow \) Denote solution by ...
  \[ \begin{align*}
    x &= \Psi(t) \\
    y &= \Psi(t)
    \end{align*} \]

  - If \( F(x, y), G(x, y) \) are continuously differentiable (\( F_x, F_y, G_x, G_y \) all continuous) near \((x_0, y_0)\), then the solution to the IVP exist near \( t = t_0 \). (Existence & Uniqueness Theorem)

  The solution \( \{ x = \Psi(t), y = \Psi(t) \} \) gives a curve over the \( xy \)-plane. Although we cannot figure our \( \Psi(t) \), \( \Psi'(t) \) explicitly, the derivatives \( \Psi'(t), \Psi''(t) \) at point \( t = t_0 \) can be seen from ODEs.

  \[ \begin{align*}
    \Psi'(t) &= F(\Psi(t_0), \Psi(t_0)) = F(x_0, y_0) \\
    \Psi''(t) &= G(\Psi(t_0), \Psi(t_0)) = G(x_0, y_0)
    \end{align*} \]

  \( \Rightarrow \) i.e., the tangent vector of the curve at \((x_0, y_0)\) can be seen. Moreover, note that this tangent vector does not depend on choice of \( t_0 \). This yields the following observation:

  For any solution of the system, if its curve over the \( xy \)-plane passes through \((x_0, y_0)\)
then the tangent vector of the curve at \((x_0, y_0)\) is \[
\begin{bmatrix}
F(x_0, y_0) \\
G(x_0, y_0)
\end{bmatrix}
\]

*Direction Field*: the vector field are in the \(xy\)-plane, or "phase plane," specified by
\[
(x_0, y_0) \rightarrow \begin{bmatrix} F(x_0, y_0) \\ G(x_0, y_0) \end{bmatrix}
\]
This can **ALWAYS** be drawn!

*Integral Curve*: (Trajectory/Orbit) the curve given by one solution to the system.

*Phase Portrait*: Phase plane + representation set of different integral curves.

**Linear Case**
\[
\begin{align*}
x' &= a_{11} x + a_{12} y \\
y' &= a_{21} x + a_{22} y
\end{align*}
\]
\Rightarrow \lambda_1, \lambda_2 \text{ denote the eigenvalues of } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}

1. **Case \#1**: Real, Distinct Eigenvalues
   a. \(\lambda_1 > \lambda_2 > 0\) \(\Rightarrow\) Unstable
   b. \(0 > \lambda_1 > \lambda_2\) \(\Rightarrow\) Stable (asymptotically stable)

\[
\vec{x}(t) = C_1 e^{\lambda_1 t} \vec{u} + C_2 e^{\lambda_2 t} \vec{v}
\]

Equilibrium = Nonlinear Source

\(\Rightarrow\) Unstable

Equilibrium = Nonlinear Sink

\(\Rightarrow\) Stable
2 Case #2: Real, Repeated Eigenvalues

(a) $\lambda_1 = \lambda_2 = \lambda > 0$

\[ \dot{x}(t) = C_1 e^{\lambda t} \dot{v} + C_2 e^{\lambda t} (t \dot{v} + \ddot{w}) \]

(b) $\lambda_1 = \lambda_2 = \lambda < 0$

\[ C_3 > 0 \]
\[ C_3 < 0 \]

Asymptotically Stable

Unstable

Improper Node

(c) $\lambda_1 = \lambda_2 = \lambda = [\lambda \ \lambda]$

\[ \text{with 2 independent eigenvectors} \]

(i) $\lambda > 0$

(ii) $\lambda < 0$

Proper Nodes

If $y(t) = x(t)$

\[ \dot{x}(t) = C_1 e^{\lambda t} \dot{v} + C_2 e^{\lambda t} \ddot{w} t \]
→ Degenerate cases (some eigenvalues being 0) is not used in nonlinear systems

⇒ We always consider non-zero eigenvalues!

Case #3: Complex Eigenvalue Cases

By an argument involving polar coordinate transformation, phase portraits look like...

(a) ξ > 0 (Reλ > 0)

⇒ Spiral Source

UNSTABLE

\( \dot{x}(t) = e^{\xi t} (c_1 \cos \beta t + c_2 \sin \beta t) \)

(b) ξ < 0 (Reλ < 0)

⇒ Spiral Sink

ASYMPTOTICALLY STABLE

(c) ξ = 0 (Reλ = 0)

⇒ Center Point

STABLE

Example: \( \dot{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \dot{x} \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 + 1 = 0 \)

⇒ \( \lambda = 2 \pm i \)

⇒ Reλ = 2 > 0

Therefore, we will get Spiral Source.

**Orientation can be obtained by testing the derivative at one arbitrary point.

At point (1,0), tangent vector

\[ \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \]
Example: \( \dot{x}^1 = \begin{bmatrix} -2 & 13 \\ -1 & 2 \end{bmatrix} \dot{x} \Rightarrow \lambda = \pm 3i \Rightarrow \text{Re} \lambda = 0 \) \( \Rightarrow \text{Center Point} \)

Tangent vector of integral curve at \((1, 0)\)

\[
\begin{bmatrix} -2 & 13 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \end{bmatrix}
\]

• We will use the phase portraits for the linear systems to investigate general nonlinear systems.

For \( \begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases} \) Critical Point: \((x_0, y_0)\)

such that \( \begin{cases} F(x_0, y_0) = 0 \\ G(x_0, y_0) = 0 \end{cases} \)

• If the initial condition is chosen at a critical point, then the solution is constantly \( \begin{cases} x = x_0 \\ y = y_0 \end{cases} \)

Example: \( \begin{cases} x' = (x+2)(y-1) \\ y' = x(y-2) \end{cases} \)

\( \Rightarrow \) To look for critical points, set

\( \begin{cases} (x+2)(y-1) = 0 \\ x(y-2) = 0 \end{cases} \)

\( \Rightarrow 1\text{st} \) Equation: \( x = -2 \) or \( y = 1 \)

If \( x = -2 \), the 2\text{nd} equation...

\( \Rightarrow -2(y-2) = 0 \)

\( \Rightarrow y = 2 \)

If \( y = 1 \), the 2\text{nd} equation...

\( \Rightarrow x(-3) = 0 \Rightarrow x = 0 \)
$\Rightarrow$ Critical Points: $(-2,2), (0,1)$

- [Locally] Linear System:

At each critical point $(x_0, y_0)$, the Jacobian matrix $J(x_0, y_0) = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}$ is nondegenerate (meaning $\det \neq 0$), plus other requirements that you will not understand until later classes.

* Recall: $f(x) = f(x_0) + f(x_0)(x-x_0) + O((x-x_0)^2)$

$$ \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F(x_0,y_0) \\ G(x_0,y_0) \end{bmatrix} + \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + O((x-x_0)^2 + (y-y_0)^2) $$

Multi-variable generalization of linear approx:

$\Rightarrow$ For locally linear systems near critical point, we use the linear system $\hat{x}' = J(x_0, y_0) \hat{x}$ to approx. the solution

\begin{align*}
(1-y)(-\rho x) = (x-x_0) \\
(-\rho y + 1)(-\rho x) = (y-y_0)
\end{align*}