Recall: Second order linear homogeneous ODE with constant coefficients:
\[ ay'' + by' + cy = 0, \quad a, b, c \text{ real numbers} \]

Characteristic equation: \[ ar^2 + br + c = 0 \]

Characteristic roots: \( r_1, r_2 \)

Case I: \( r_1, r_2 \) real and distinct.

General solution: \[ y = C_1 e^{rt} + C_2 e^{rt} \]

Case II: \( r_1 = r_2 = r \) repeated (automatically real)

General solution: \[ y = C_1 e^{rt} + C_2 te^{rt} \]

Case III: \( r_1 \neq r_2 \) complex. Write \( r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta \)

General solution: \[ y = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t \]

Leftovers for the previous class: repeated root case. How comes the \( t \) factor in the other solution.

Variation of Parameters

Philosophy: If you know one solution \( y_1 \), then since the ODE is linear, \( cy_1 \) will also be a solution for any number \( c \) parameter.

Change this numeric parameter into a function, i.e., set \( y_2(t) = u(t) \cdot y_1(t) \), then plug it into the ODE to find \( u(t) \).
1 Application to homogeneous ODE.

Knowing $y_1$ is a solution to

$$y'' + p(t)y' + q(t)y = 0$$

Set $y_2(t) = u(t)y_1(t)$, put it back to the ODE.

⇒ another ODE with order reduced by one that leads to $u(t)$.

$$y_2'' + p y_2' + q y_2 = (u y_1)'' + p (u y_1)' + q (u y_1)$$

$$(u y_1)' = u' y_1 + u y_1', \ (u y_1)'' = u'' y_1 + 2 u' y_1' + u y_1$$

$$= u'' y_1 + 2 u' y_1' + u y_1' + p (u y_1' + u y_1) + q u y_1$$

$$= u'' y_1 + u' (2 y_1' + p y_1) + u' (y_1'' + p y_1' + q y_1)$$

$y_1$ is a soln. ⇒ $y_1'' + p y_1' + q y_1 = 0$

$$= u'' y_1 + u' (2 y_1' + p y_1) = 0 \ (we \ set \ y_2 \ as \ a \ soln.)$$

In other words, if $y_2 = u y_1$ is a solution, then $u$ must satisfy

$$y_1 u'' + (2 y_1' + p y_1) u' = 0$$

This can be regarded as a first order ODE concerning $u'$.

More precisely, set $v = u'$, then

$$y_1 v' + (2 y_1' + p y_1) v = 0$$

We can solve $v$ from this ODE ⇒ $u$ (by integration)

⇒ $y_2$ (by multiplying $u$ to $y_1$) ⇒ Gen. soln: $y = C_1 y_1 + C_2 y_2$. 
Example: \( y'' - 2ry' + ry = 0 \). \( r \) real number.

Know from char. eqn. that \( y = e^{rt} \) is a sol'n.

Set \( y_1 = u(t) \), \( y_1(t) = u(t) e^{rt} \). We know from above that \( u \) satisfies
\[
y_1 u'' + (2ry_1' + py_1) u' = 0
\]
\[
e^{rt} u'' + (2re^{rt} - 2re^{rt}) u' = 0 \Rightarrow e^{rt} u'' = 0 \Rightarrow u'' = 0
\]

Integrate: \( u' = C_1 \)

Integrate again: \( u = C_1 t + C_2 \)

\[
y_2 = u y_1 = (C_1 t + C_2) e^{rt} = C_1 t e^{rt} + C_2 e^{rt}
\]

This means \( C_1 t e^{rt} + C_2 e^{rt} \) will be another sol'n for any \( C_1, C_2 \).

When \( C_1 = 0, C_2 = 1 \), we recover \( y_1 \).

\[
y''(t) y_1(t) \neq 0 \]

\( \Rightarrow \) Gen. sol'n \( y = C_1 t e^{rt} + C_2 e^{rt} \).

Remarks: 1. Normally when solving for \( u(t) \), we normally will set these
arbitrary constants as concrete numbers so as to simplify the
computation. However, if you don't do that, then \( y = uy_1 \)
will give the general sol'n.

2. When formulating the ODE concerning \( u \), make sure your \( p \)
comes from the standard form. Also notice that \( q \) is not used.
Example: \( ty'' - y' - 4t^3 y = 0 \). Knowing \( y_1 = \sin(t^3) \) is a solution, find the general solution.

\[
\text{Std. form: } y'' - \frac{1}{t} y' - 4t^3 y = 0.
\]

Set \( y_2 = uy_1 \). \( y_1 = \sin(t^3) \). \( y_1' = 2t \cos(t^3) \quad p = -\frac{1}{t} \)

\[
\sin(t^3) \cdot u'' + \left( 4t \cos(t^3) - \frac{1}{t} \sin(t^3) \right) u' = 0.
\]

\[
\frac{u''}{u'} = \frac{4t \cos(t^3) - \frac{1}{t} \sin(t^3)}{-\sin(t^3)}
\]

\[
= -\frac{4t \cos(t^3)}{-\sin(t^3)} + \frac{1}{t}
\]

Integrate: \( \ln|u'| = \ln|t| - \int \frac{4t \cos(t^3)}{-\sin(t^3)} \, dt \)

\[
\int \frac{4t \cos(t^3) \, dt}{-\sin(t^3)} \frac{u = \sin(t^3)}{du = 3t^2 \cos(t^3) \, dt} \quad \int \frac{2du}{u} = 2 \ln|u| + C = 2 \ln\left| \sin(t^3) \right| + C
\]

\[
= \ln|t| - 2 \ln|\sin(t^3)| \quad \text{took } C = 0
\]

\[
\ln|u'| = \ln\left| \frac{t}{\sin^2(t^3)} \right|
\]

\[
\ln a - \ln b = \ln \frac{a}{b} \quad , \quad C \ln a = \ln a^C
\]

\[
u' = \frac{t}{-\sin^2(t^3)}
\]

Integrate again: \( u = \int \frac{t}{-\sin^2(t^3)} \, dt \)

\[
= -\frac{1}{2} \int \csc(t^3) \cdot d(t^3) \quad \int \csc^2 t = -\cot t + C.
\]

\[
= -\frac{1}{2} \cot(t^3)
\]

\[
\frac{1}{\sin^2 t} = \csc^2 t.
\]

again we don't care about \( C \).
\[ y_2 = u y_1 = -\frac{1}{2} \cot(t^3) \cdot \sin(t^3) = -\frac{1}{2} \cos(t^3) \]

General solution: \[ y = C_1 \sin(t^3) + C_2 (-\frac{1}{2} \cos(t^3)) \]

\[ = C_1 \sin(t^3) + C_2 \cos(t^3) \].

Free Vibrations.

A mass is attached with a spring vertically. The mass is subject to:

1. Gravity = mg
2. Force of the spring = \( k \Delta x \) \( \Delta x = \) displacement of the spring from the natural length (Hooke’s law)
3. Damping force = \( \gamma u' \) with direction opposite to the direction of motion and being proportional to the velocity.

Let \( u \) be the displacement of the mass from the equilibrium. Take downside to be the positive direction.

\[ m \frac{d^2u}{dt^2} = mg - k \Delta x - \gamma u' \]

Since \( mg = kl \), \( mg - k \Delta x = kl - k \Delta x = - k(\Delta x - l) = -ku \).

\[ \Rightarrow mu'' = -ku - \gamma u' \Rightarrow mu'' + \gamma u' + ku = 0. \]
Case 1: Undamped case: $\gamma = 0$.

The ODE becomes: $mu'' + ku = 0$

Char. eqn: $mr^2 + k = 0 \Rightarrow r^2 = -\frac{k}{m} \Rightarrow r = \pm \sqrt{\frac{k}{m}} i$

Gen. soln: $u = C_1 \cos\left(\sqrt{\frac{k}{m}} t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}} t\right)$

Write $\omega = \sqrt{\frac{k}{m}}$. Gen. soln = $C_1 \cos \omega t + C_2 \sin \omega t$.

With initial value specified, we can solve $C_1$, $C_2$ as concrete numbers.

$u = C_1 \cos \omega t + C_2 \sin \omega t$

$= \sqrt{C_1^2 + C_2^2} \cos(\omega t - \phi)$

Natural frequency: $\omega = \sqrt{\frac{k}{m}}$ Period: $\frac{2\pi}{\omega}$

Amplitude: $A = \sqrt{C_1^2 + C_2^2}$

Phase: $\phi$, determined by the angle of $(C_1, C_2)$ on the plane.

Example: $mg = 10$ lb, $10$ lb = $k \cdot 2$ in. = $k \cdot \frac{1}{6}$ ft.

$u(0) = 2$ in $= \frac{1}{6}$ ft $u'(0) = -1$ ft/s.

ODE: $mu'' + ku = 0$ $m = \frac{10 \text{lb}}{32 \text{ ft/s}^2} = \frac{5}{8} \text{ lb} \cdot \text{s}^2/\text{ft}$, $k = 60 \text{ lb}/\text{ft}$.

$\frac{5}{8} u'' + 60u = 0 \Rightarrow u'' + 96u = 0$, $u(0) = \frac{1}{6}$, $u'(0) = -1$.

Char. roots: $r = \pm \sqrt{96} i = \pm 4\sqrt{6} i$

Gen. soln: $u = C_1 \cos(4\sqrt{6} t) + C_2 \sin(4\sqrt{6} t)$

$u(0) = \frac{1}{6} \Rightarrow C_1 = \frac{1}{6}$. $u'(0) = -1 \Rightarrow 4\sqrt{6} C_2 = -1 \Rightarrow C_2 = -\frac{1}{4\sqrt{6}}$. 
Solution: \( u = \frac{1}{6} \cos(4\sqrt{6}t) - \frac{1}{4\sqrt{6}} \sin(4\sqrt{6}t) \).

Amplitude: \( \sqrt{\frac{1}{36} + \frac{1}{96}} = \sqrt{\frac{1}{6 \times 6} + \frac{1}{16 \times 6}} = \sqrt{\frac{1}{12} \left( \frac{1}{3} + \frac{1}{8} \right)} = \sqrt{\frac{1}{12} \cdot \frac{11}{12 \times 2}} = \frac{1}{\sqrt{12 \times 2}} \) stays constant = steady oscillation.

Natural frequency = 4\sqrt{6}. Period = \( \frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}} \).

Phase: \( \varphi = \arctan \frac{6}{4\sqrt{6}} = \arctan \frac{3}{2\sqrt{6}} = \arctan \frac{\sqrt{6}}{4} \).

HW. Skip 3, 4, 5, 1a.

Attendance Quiz: Knowing \( y_1 = \frac{1}{t} \) is the solution of \( t^2 y'' + 3ty' + y = 0 \), find the general solution.
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LECTURE NOTES OF DIFFERENTIAL EQUATION