Introduction to Differential Equations.

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Differential Equation: Equation concerning derivatives.

Typical first order ordinary differential equation (ODE)

\[ y' = f(t, y) \quad y'(t) = f(t, y(t)) \]

- \( t \) variable
- \( y = y(t) \) one variable function depends on \( t \)
- \( y' = \frac{dy}{dt} \) derivative of \( y \)
- \( f(t, y) \) known function of two variables

This ODE expresses \( y' \) in terms of \( t, y(t) \)
General form of a first order ODE
\[ F(t, y, y') = 0 \]

General form of an \( n \)-th order ODE
\[ F(t, y, y', \ldots, y^{(n)}) = 0 \]

Examples:
\[
\begin{align*}
y' &= t^2 + y \\
(y')^3 + \sin y &= y + t^2 \\
y'' + 3y' + 3y + y &= e^{2t}
\end{align*}
\]

3rd order

Why study ODEs?
ODE is widely used in real life.

Example: Newton's law of cooling:

Object placed in a room.
Temperature of the object changes, the rate of change is proportional to the difference of the temperature between the obj. & the room.
\[ \frac{dT}{dt} = -k(T - T_a) \]

Example: Lake, volume of water = \( V \) m\(^3\)

Assume \( V \) is constant

A factory emits mercury into the lake with a rate \( R \) kg/day

Suppose water refreshes every day by \( W \) m\(^3\)/day

How much time it takes for the water to be unpotable.

Mass of Hg at time \( t \) — \( P(t) \)

Take a small time period \( \Delta t \)

Within \( \Delta t \):

- Mercury in: \( R \Delta t \)
- Polluted water out: \( W \Delta t \)

Density: \( \frac{P(t)}{V} \)
Mercury out: \( \frac{P(t)}{V} W \Delta t \)

Thus the change of mercury \( \Delta P \) within \( \Delta t \):
\[
\Delta P = R \Delta t - \frac{P(t)}{V} W \Delta t
\]

Dividing \( \Delta t \) both sides:
\[
\frac{\Delta P}{\Delta t} = R - \frac{P(t)}{V} W
\]

Taking \( \Delta t \to 0 \):
\[
\frac{dP}{dt} = R - \frac{P}{V} W
\]

EPA std: not potable when \( \frac{P(t)}{V} < 0.002 \text{ mg/ft}^3 \)

Assume \( P(0) = 0 \). We're looking for \( T \) s.t. \( \frac{P(T)}{V} \) arrives the threshold.
Solution of Ex. 1: assume $T_a$ constant

$$\frac{dT}{dt} = -k \ (T - T_a)$$

Separating Variables:

$$\frac{dT}{T - T_a} = -k \ dt$$

Integrate both sides:

$LHS = \int \frac{dT}{T - T_a} = \ln |T - T_a| + C$

$RHS = \int -kd\ t = -kt + C'$

$\Rightarrow \ln |T - T_a| = -kt + C$

Exponentiate:

$$T - T_a = e^{-kt + C} = e^{-kt} e^C = C e^{-kt}$$

$[why \ I\ can\ drop\ abs.\ val?]$

$$T = T_a + C \cdot e^{-kt}.$$

Notice: $C$ is arbitrary. Without further info specified

there's no way to decide $C$.

In other words, $T(t) = T_a + C e^{-kt}$ are solutions to
the ODE for ANY $C$. 
Such solns with arbitrary constants are referred as General Solutions.

\[ ODE \Leftrightarrow \text{Gen. soln} \]

Solution to Example 2:

\[
\frac{dP}{dt} = R - \frac{PW}{V} = R - \frac{W}{V} P
\]

Separate variables:

\[
\frac{dP}{R - \frac{WP}{V}} = dt.
\]

Integrate both sides:

\[
\text{LHS} = \int \frac{dP}{R - \frac{WP}{V}} = -\frac{V}{W} \ln |R - \frac{WP}{V}| + C
\]

\[
\int \frac{dP}{R - \frac{WP}{V}} = \int -\frac{V}{W} \cdot \left( -\frac{W}{V} \right) dP = -\frac{V}{W} \int \frac{d(-\frac{WP}{V})}{R - \frac{WP}{V}}
\]

\[
= -\frac{V}{W} \cdot \ln |R - \frac{WP}{V}| + C
\]

\[
\text{RHS} = t + C'
\]

\[-\frac{V}{W} \ln |R - \frac{WP}{V}| = t + C
\]
\[ \ln |R - \frac{W}{V}P| = -\frac{W}{V}t + c \]

\[ R - \frac{W}{V}P = Ce^{-\frac{W}{V}t} \]

\[ P = \frac{V}{W}(R - Ce^{-\frac{W}{V}t}) \]

By assumption, \( P(0) = 0 \).

\[ P(0) = \frac{V}{W}(R - C \cdot e^0) = 0 \]

\[ \Rightarrow C = R \]

\[ P(t) = \frac{VR}{W}(1 - e^{-\frac{W}{V}t}) \]

The arbitrary constant is determined by the initial value!

**Initial Value Problem (IVP) = ODE + initial value**

An IVP has a unique solution.

**Summary:** ODE (first order) \( \rightarrow \) General Solution

IVP (first order) \( \rightarrow \) Solution
Not all ODE can be solved in the above way!
In general, we can't expect to solve all the ODEs.
But there is a way to get some information of the sol'n.
this way works for every ODE.

**Geometric Interpretation**

\[
\begin{aligned}
  & y' = f(t, y) \\
  & y(t_0) = y_0
\end{aligned}
\]

Put \( t=t_0 \) to the ODE

\[
  \Rightarrow y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0)
\]

This tells that the derivative of solution to the IVP at \( t=t_0 \) is known.

**Example:** \( y' = 2y - 1 \), \( y(2) = 3 \).

\[
  y'(2) = 2 \times 3 - 1 = 5.
\]

Collecting all the derivatives for every pt in \( t-y \)-plane, use
line elements to denote the derivative, we got a picture called direction field.

Example: \( y' = 2y - 1 \)

RHS indep. of \( t \).

\[
\begin{array}{c|c}
\text{\( y_0 \)} & f(t_0, y_0) \\
\hline
2 & 3 \\
3/2 & 2 \\
1 & 1 \\
1/2 & 0 \\
0 & -1 \\
-1/2 & -2 \\
-1 & -3 \\
\end{array}
\]

You should know how to draw the direction field for \( y' = f(y) \), (RHS not involves \( t \))

From direction field, we'll learn at least where the solution goes: if \( y > 0 \), sol'n ↑

\( y' = 0 \) sol'n stays

\( y' < 0 \) sol'n ↓
For the example $y' = 2y - 1$, if $(t_0, y_0)$ lies above $y = \frac{1}{2}$ then the solution increases, going away from $\frac{1}{2}$.

If $(t_0, y_0)$ lies on $y = \frac{1}{2}$, then $y$ stays $= \frac{1}{2}$.

If $(t_0, y_0)$ lies below $y = \frac{1}{2}$, then $y$ goes away from $\frac{1}{2}$.

Example: $y' = (y + 1)(y - 2)$.

<table>
<thead>
<tr>
<th>$y_0$</th>
<th>$y'(t_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
</tbody>
</table>

The actual solution, represented in the $yt$-plane, are called "integral curve" (trajectory of the solution).

Direction field in green. Integral curve in purple.

Attendance quiz: Draw the dir. field for $y' = y(y - 4)$ make sure $y = 4$ and $y = 0$ are included.