2.5. Summary:

(1) Autonomous ODE: $y' = f(y)$.

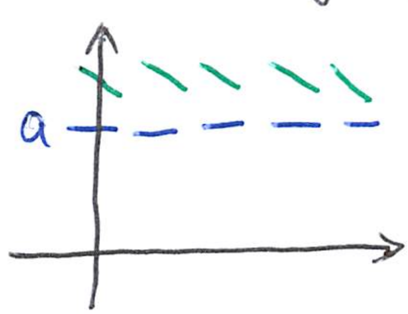
RHS depends only on y, not on x.

(2) Slopes in the direction field stays the same if only x is changed.

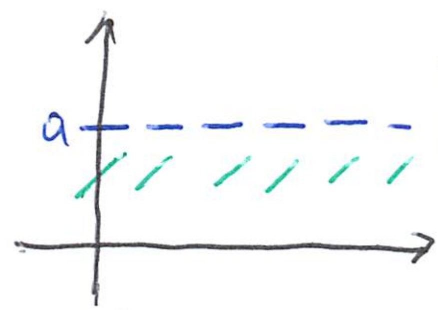
(3) Equilibrium: $y = a$ ~~s.t.~~ the constant solution

To find it, ~~is~~ solve $f(y) = 0$.

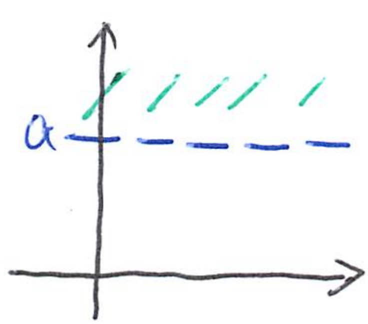
(4). Stability of equilibrium.



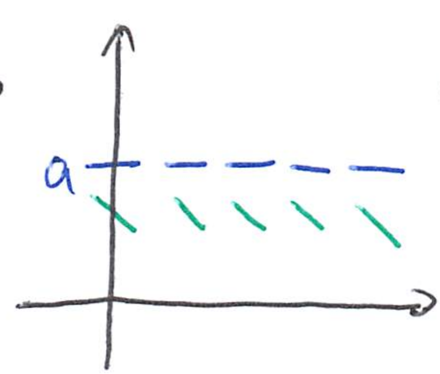
stable from above.



stable from below.



unstable from above



unstable from below.

If ~~a~~ stable from both directions, then $y=a$ is called (asymptotically) stable.

If unstable from both directions, then $y=a$ is called (asymptotically) unstable

If stable from one direction, unstable from the ~~another~~ other, then $y=a$ is called (asymptotically) semistable.

(5) To determine stability, it suffices to know the sign of $f(y)$ near the ~~sta~~ ~~equilib~~ equilibrium.

It's certainly better to know its graph. Sometimes drawing the graph is easier (especially ~~if~~ when you have a computer ~~in hand~~ around).

Rmk: The book analyzes phase lines.

I taught how to find signs in class.

Dr. Z's note gives a numerical approach.

Use whatever is convenient.

Example: Book Problem 2.5.2.

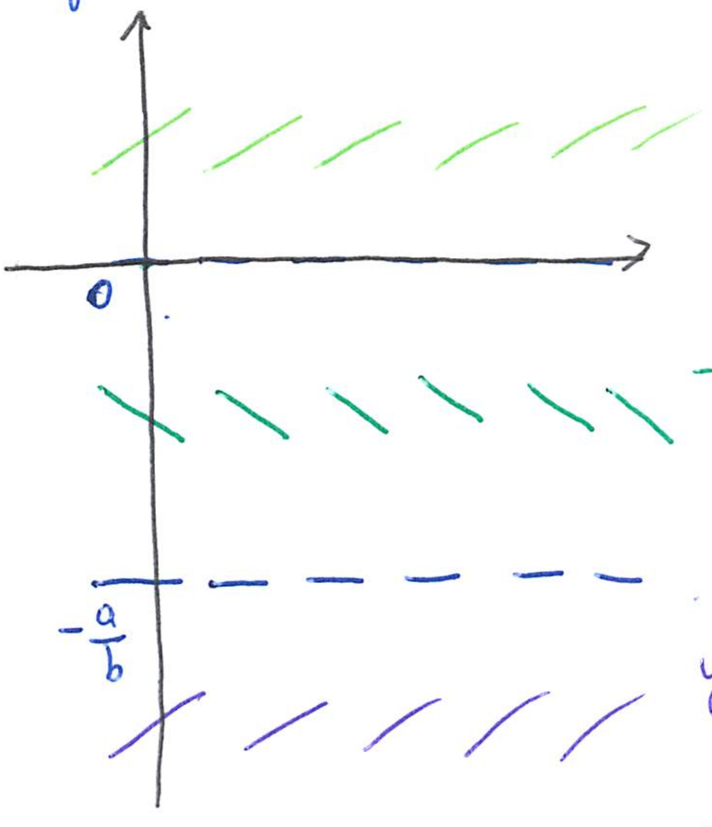
$$\frac{dy}{dt} = ay + by^2, \quad a>0,\ b>0,\ -\infty < y_0 < \infty$$

Find equilibriums:

$$ay+by^2=0 \Rightarrow (a+by)\,y = 0 \Rightarrow y=0 \text{ or } y=-\frac{a}{b}.$$

To find stability:

Way 1: Draw direction field.



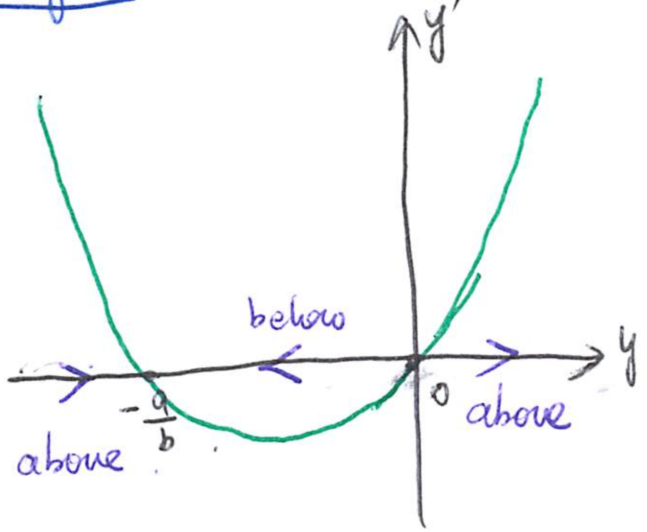
$y>0$, $\underset{+}{(a+by)}\,\underset{+}{y} > 0$.

$-\frac{a}{b} < y < 0$ $\underset{+}{(a+by)}\,\underset{-}{y} < 0$

$y < -\frac{a}{b}$ $\underset{-}{(a+by)}\,\underset{-}{y} > 0$.

Conclusion: $y=0$ ~~sta~~ unstable. $y=-\frac{a}{b}$ stable.

Way 2: Phase line.



1. Draw $f(y) = ay + by^2$.

2. Draw arrows on y-axis according to the graph.

Above y-axis: $\rightarrow$

Below y-axis: $\leftarrow$.

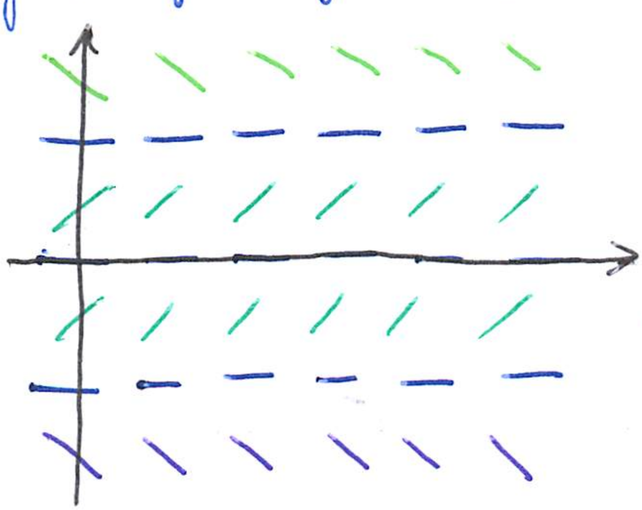
Conclusion: $y = -\frac{a}{b}$ stable, $y = 0$ unstable.

Example: $$\frac{dy}{dt} = y^2(4 - y^2).$$

Find equilibriums:

double root.

$$y^2(4-y^2) = y^2(2-y)(2+y) = 0 \Rightarrow y = -2, 0, 0, 2.$$



$y > 2$ $\quad y^2(2-y)(2+y) < 0.$ $\quad + \; - \; +$

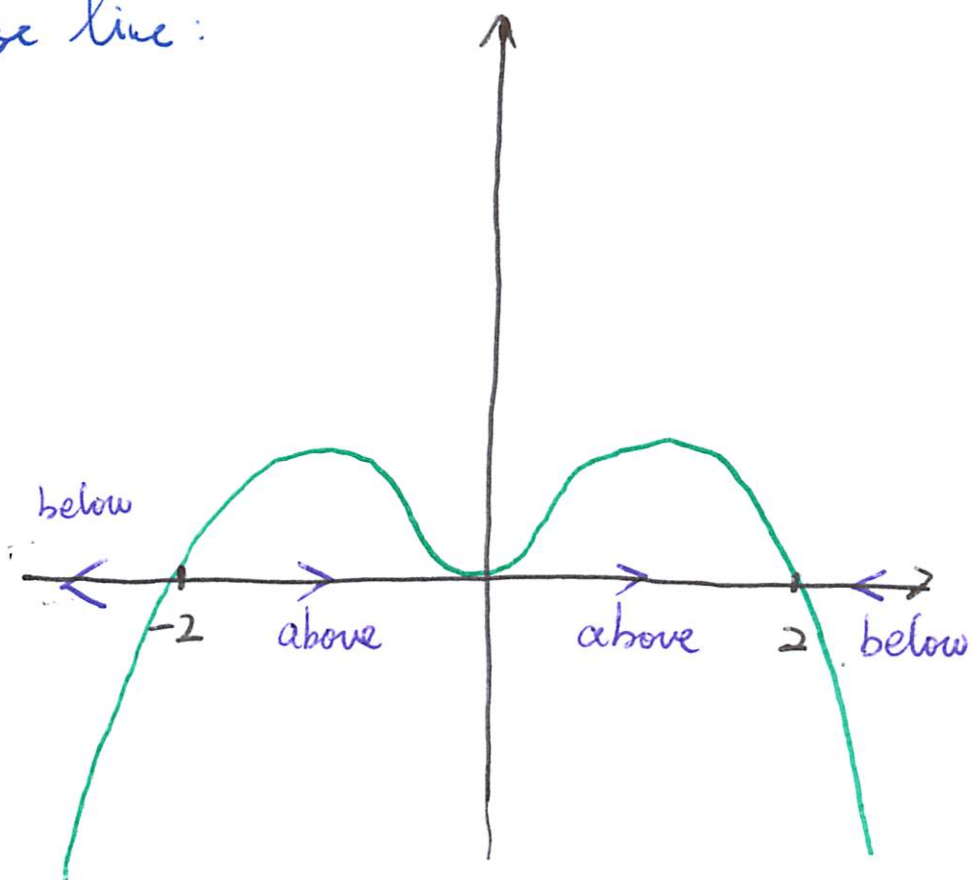
$0 < y < 2$ $\quad y^2(2-y)(2+y) > 0.$ $\quad + \; + \; +$

$-2 < y < 0$ $\quad y^2(2-y)(2+y) > 0$ $\quad + \; + \; +$

$y < -2$ $\quad y^2(2-y)(2+y) < 0$ $\quad + \; + \; -$

Conclusion: $y=2$ stable, $y=0$ semistable, $y=-2$ unstable.

Phase line:



Draw $f(y) = y^2(2-y)(2+y)$.

Notice $\lim_{y\to\pm\infty} f(y) = -\infty$.

∴ graph starts in southwest, traverse $(-2,0)$, tangent to $(0,0)$, traverse $(2,0)$, end in southeast.

Draw arrows on y-axis: ABOVE $\Rightarrow$ $\rightarrow$.

BELOW $\Rightarrow$ $\leftarrow$.

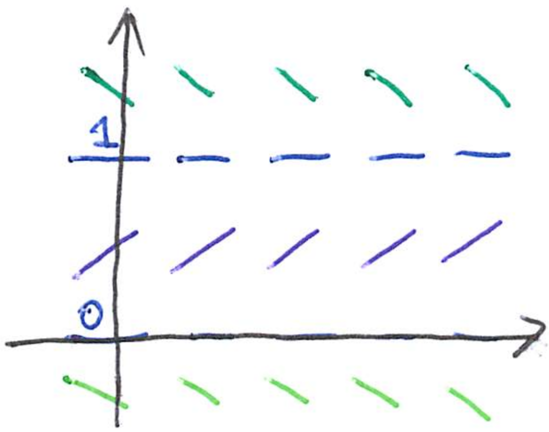
Conclusion: $y=-2$ unstable, $y=0$ semistable, $y=2$ stable.

Example: ~~2~~. Book Problem 2.5.22.

$$\frac{dy}{dt} = \alpha(1-y)y. \quad y(0) = y_0. \quad \alpha > 0$$

Equilibrium: $y = 0, \; y = 1$.

Stability:



stable

unstable.

Actual solution: separate variables:

$$\frac{dy}{y(1-y)} = \alpha \, dt.$$

Recall partial fractions: $\displaystyle\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$.

So $\displaystyle\int \frac{dy}{y(1-y)} = \int \frac{dy}{y} + \int \frac{dy}{1-y} = \ln|y| - \ln|1-y|$

$$= \int \alpha \, dt = \alpha t + C.$$

Recall $\ln a - \ln b = \ln \frac{a}{b}$:

So $\ln|y| - \ln|1-y| = \alpha t + C$

$\Rightarrow \ln\left|\frac{y}{1-y}\right| = \alpha t + C$

$\Rightarrow \frac{y}{1-y} = Ce^{\alpha t}$. Put $t=0 \Rightarrow \frac{y_0}{1-y_0} = C$.

Solve y : $y = \frac{Ce^{\alpha t}}{1+Ce^{\alpha t}} = \frac{\frac{y_0}{1-y_0}e^{\alpha t}}{1+\frac{y_0}{1-y_0}e^{\alpha t}}$

$$= \frac{y_0 e^{\alpha t}}{1-y_0+y_0e^{\alpha t}}.$$

$$= \frac{y_0}{(1-y_0)e^{-\alpha t}+y_0}.$$

Discuss y_0 :

$y_0 > 1$, $y(t) = \frac{y_0}{(1-y_0)e^{-\alpha t}+y_0} \to \frac{y_0}{0+y_0} = 1$. $\alpha > 0$ $e^{-\alpha t} \to 0$ as $t \to \infty$.

$y_0 = 1$. $y(t) = \frac{y_0}{0+y_0} = 1$.

$0 < y_0 < 1$ $y(t) = \frac{y_0}{(1-y_0)e^{-\alpha t}+y_0} \to \frac{y_0}{0+y_0} = 1$.

(What I messed up in class)

$y_0 < 0$. $\quad y(t) = \dfrac{y_0}{\underset{\substack{\uparrow \\ \text{positive}}}{(1-y_0)e^{-\alpha t}} + \underset{\substack{\uparrow \\ \text{negative}}}{y_0}}$ does not extends to $(0, +\infty)$.

In fact, the solution exists whenever $(1-y_0)e^{-\alpha t} + y_0 \neq 0$

i.e. $e^{-\alpha t} \neq \dfrac{y_0}{y_0 - 1}$. $\quad t \neq -\dfrac{1}{\alpha} \ln \dfrac{y_0}{y_0-1}$.

$y_0 > y_0 - 1$

$$-\frac{1}{\alpha}\ln\frac{y_0}{y_0-1} = \frac{1}{\alpha}\ln\frac{y_0-1}{y_0}.$$

$1 < \dfrac{y_0-1}{y_0}$ b/c $y_0 < 0$.

$\Downarrow$

$0 < \ln \dfrac{y_0-1}{y_0}$.

i.e. this is a positive number.

Since the initial values takes at $t=0$, the solution exists only in the interval

$$0 \leqslant t < \frac{1}{\alpha}\ln\frac{y_0-1}{y_0}.$$

As $t \to \dfrac{1}{\alpha}\ln\dfrac{y_0-1}{y_0}$, $\quad y(t) \to -\infty$.

Rmk: Nevertheless the model didn't ask $y_0 > 1$ or $y_0 < 0$.
I'm just covering my own ass on math.

2.6 Summary:

(1) Exact ODE: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

such that there exists a function $\Psi(x,y)$ with

$$\frac{\partial \Psi}{\partial x} = M(x,y), \quad \frac{\partial \Psi}{\partial y} = N(x,y).$$

In this case, the equation

$$\Psi(x,y) = C$$

gives an implicit ~~solution~~ solution to $M + Ny' = 0$.

More over, write $y = y(x)$ the function determined by $\Psi(x,y) = 0$. One should have

$$\frac{d}{dx}\Psi(x,y(x)) = M(x,y(x)) + N(x,y(x))\frac{dy}{dx}$$

Recall: $\frac{d}{dx}\Psi(x,y(x)) = \frac{\partial \Psi}{\partial x}(x,y(x)) + \frac{\partial \Psi}{\partial y}(x,y(x))\cdot\frac{dy}{dx}$.

(2). Equivalent condition:

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. or in short, $M_y = N_x$.

(3) How to recover ψ?

Way 1: Integrate M:

$$\psi(x,y) = \int M(x,y)\,dx + \varphi(y).$$

b/c the integral's on x. Any func. on y may serve as the undetermined φ term.

Take the partial derivative $\frac{\partial}{\partial y}$ to whatever you obtain, set it equal to $N(x,y)$, find $\varphi'(y)$ and integrate it, to get $\varphi(y)$, and thus $\psi(x,y)$. Or in formula:

$$\frac{\partial \psi}{\partial y} = N(x,y) \xRightarrow{\text{solve}} \varphi'(y) \xRightarrow{\text{int.}} \varphi(y) \Rightarrow \psi(x,y).$$

Remark: If the $\varphi'(y)$ is not independent x, then you must have messed up somewhere. Don't you dare to continue!

Remark: When integrate $\varphi'(y)$, we never care about the constant, as it will be absorbed by the C on the RHS of ~~the~~ $\psi(x,y) = C$.

Way 2: Compute the integral of N.

$$\Psi(x,y) = \int N(x,y)\,dy + \varphi(x).$$

Solve $\varphi'(x)$ from $\frac{\partial \Psi}{\partial x} = M(x,y)$ and integrate to get $\varphi(y)$ and thus $\Psi(x,y)$.

WARNING: Although recovering $\Psi(x,y)$ takes most of the labor, $\Psi(x,y)$ itself is NOT a solution! Always present the solution as $\Psi(x,y) = C$.

(4) How to check?

Make use of the fact $\frac{d}{dx}\Psi(x,y(x)) = M(x,y(x)) + N(x,y(x))y'$

More precisely, apply $\frac{d}{dx}$ to the equation $\Psi(x,y) = C$ (regarding y as a function of x). If the sol'n is correct, you should recover the original ~~ODE~~ exact ODE.

(5). What if the ODE is not exact?

There is NO general way to deal with such cases

However if $\frac{M_y - N_x}{N}$ happens to be independent of y ~~or~~, i.e. is a function depending ONLY on x, then by solving $\frac{\mu'(x)}{\mu(x)} = \frac{M_y - N_x}{N}$, one obtains an integrating factor $\mu(x)$. Multiply $\mu(x)$ to your ODE, you should get a NEW, EXACT ODE, which can be solved and the solution ~~gives~~ ~~is~~ satisfies your original ODE.

Or if $\frac{M_y - N_x}{-M}$ is independent of x, one can find an integrating factor by solving

$$\frac{\mu'(y)}{\mu(y)} = \frac{M_y - N_x}{-N}$$

Proceed similarly to get a solution.

Example: Book Problem 2.6.9.

$$ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x + (xe^{xy}\cos 2x - 3)y' = 0.$$

$$M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x.$$

$$N = xe^{xy}\cos 2x - 3.$$

$$M_y = \underbrace{e^{xy}\cos 2x + \cancel{xy}\,y(x\,e^{xy})\cos 2x}_{\text{derivative of first term w/ product rule}} - 2xe^{xy}\sin 2x.$$

$$N_x = e^{xy}\cos 2x + \cancel{y}\,x\,y\,e^{xy}\cos 2x + xe^{xy}(-2\sin 2x).$$

cf. $(fgh)' = f'gh + fg'h + fgh'$.

$M_y = N_x \Rightarrow$ exact!

Integrating M is intimidating. Let's integrate N:

$$\psi(x,y) = \int N(x,y)\,dy = \int\left[(xe^{xy})\cos 2x - 3\cancel{dy}\right]dy$$

$$= \cos 2x \cdot \cancel{xy}\,e^{xy} \cancel{+\varphi(x)} - 3y + \varphi(x).$$

Take $\frac{\partial}{\partial x}$ to the above:

$$\frac{\partial\psi}{\partial x} = -2\sin 2x \cdot \cancel{xy}\,e^{xy} + \cos 2x \cdot y \cdot e^{xy} + \varphi'(x)$$

Set it equal to M:

$$\frac{\partial \psi}{\partial x} = -2\sin 2x\, e^{xy} + y e^{xy}\cos 2x + \varphi'(x)$$

$$= y e^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x.$$

$$\Rightarrow \varphi'(x) = 2x \Rightarrow \varphi(x) = x^2.$$ No need to care about that constant.

$$\psi(x,y) = e^{xy}\cos 2x - 3y + x^2$$

YOU ARE NOT DONE YET!

$$\cancel{\psi}\ e^{xy}\cos 2x - 3y + x^2 = C.$$

This is the final solution!

Check: $\frac{d}{dx}\left(e^{xy}\cos 2x - 3y + x^2\right) = \frac{d}{dx} C = 0.$

$$\cancel{\text{LHS}} = \frac{d}{dx}\left(e^{xy}\right)\cos 2x + e^{xy}\frac{d}{dx}(\cos 2x) - 3\frac{dy}{dx} + 2x.$$

$$= e^{xy}\cdot(y + xy')\cos 2x - 2e^{xy}\sin 2x - 3\frac{dy}{dx} + 2x$$

$$= y e^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x + y'\left(x e^{xy}\cos 2x - 3\right) = 0.$$

precisely the ODE we started with. ✓.

Example: 2.6.26. $y' = e^{2x} + y - 1$.

Write it into $M + Ny' = 0$:

$$e^{2x} + y - 1 - y' = 0.$$

$M = e^{2x} + y - 1 \qquad N = -1.$

$M_y = 1, \quad N_x = 0. \quad M_y \neq N_x$. not exact.

Notice $\dfrac{M_y - N_x}{N} = \dfrac{1-0}{-1} = -1$ indep. of y.

Solve $\dfrac{\mu'(x)}{\mu(x)} = -1$ Notice $\frac{\mu'}{\mu} = (\ln\mu)'$.

$$\ln \mu(x) = -x \cancel{+C}$$ No need to care about C.

$$\mu(x) = e^{-x}.$$

So we have an integrating factor. Multiply!

$$e^{-x}(e^{2x} + y - 1 - y') = 0.$$

$$e^{x} + e^{-x}y - e^{-x} - e^{-x}y' = 0.$$

Set $M' = e^{x} + e^{-x}y - e^{-x}, \quad N' = -e^{-x}.$

$M'_y = e^{-x}. \quad N'_x = e^{-x}. \quad M'_y = N'_x$ exact!

Integrate N' (again it's easier).

$$\psi(x,y) = \int -e^{-x}\,dy = -e^{-x}y + \varphi(x).$$

$$\frac{\partial \psi}{\partial x} = \cancel{e^{-x}y} + \varphi'(x) = e^{+x} + \cancel{e^{-x}y} - e^{-x}$$

$$\Rightarrow \quad \varphi'(x) = e^{x} - e^{-x}.$$

$$\varphi(x) = e^{x} + e^{-x}.$$

$$\psi(x,y) = \boxed{e^{x} + e^{-x} - e^{-x}y = C}.$$

<u>Check</u>: $\frac{d}{dx}\left(e^{x} + e^{-x} - e^{-x}y\right) = 0.$

$$= e^{x} - e^{-x} \underbrace{+ e^{-x}y - e^{-x}\cdot\frac{dy}{dx}}_{\text{product rule to the 3rd term.}} = 0. \quad ✓$$