

13. FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS

We consider the parabolic problem:

$$u_t - \frac{\partial}{\partial x} \left(p \frac{\partial u}{\partial x} \right) + qu = f, \quad a < x < b, \quad 0 < t \leq T,$$

$$u(a, t) = 0, \quad u(b, t) = 0, \quad 0 < t \leq T, \quad u(x, 0) = \psi(x), \quad a < x < b.$$

Let $V^0 = \{v \in H^1(a, b) : v(a) = v(b) = 0\}$. A variational formulation of this problem is to seek $u(t) \in V^0$ such that $u(0) = \psi$ (i.e., $u(x, 0) = \psi(x)$), and

$$(\partial u / \partial t, v) + a(u, v) = (f, v), \quad v \in V^0,$$

where (\cdot, \cdot) denotes the L^2 inner product on (a, b) and now

$$a(u, v) = \int_a^b \left[p \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + quv \right] dx.$$

13.1. Continuous time Galerkin scheme. We first consider an approximation in which we discretize by finite elements in the spatial variable, but keep time continuous. Thus, we choose a finite dimensional subspace $V_h^0 \subset V^0$ and look for an approximation $u_h(t) \in V_h^0$, $t \in [0, T]$, satisfying: $u_h(0) = \psi_h$ (ψ_h an approximation to ψ) and

$$(\partial u_h / \partial t, v) + a(u_h, v) = (f, v), \quad v \in V_h^0.$$

To see what is involved in solving this problem, we write $u_h(t) = \sum_{j=1}^m \alpha_j(t) \phi_j(x)$, where $\{\phi_j\}_{j=1}^m$ is a basis for V_h^0 . Inserting this into the variational equations, and choosing v to be each of the basis functions ϕ_i , we get

$$\sum_{j=1}^m \alpha_j'(t) (\phi_j, \phi_i) + \sum_{j=1}^m \alpha_j(t) a(\phi_j, \phi_i) = (f, \phi_i), \quad i = 1, \dots, m.$$

Let

$$M_{ij} = (\phi_j, \phi_i), \quad A_{ij} = a(\phi_j, \phi_i), \quad F_i = (f, \phi_i), \quad \alpha = (\alpha_1, \dots, \alpha_m)^T.$$

Our equations then have the form

$$M\alpha'(t) + A\alpha = F,$$

a first order system of ordinary differential equations. If we write ψ_h in the form $\sum_{j=1}^m \beta_j \phi_j$, then we immediately get the initial condition that $\alpha_j(0) = \beta_j$. A simple example is when V_h is chosen to be the space of continuous piecewise linear functions on a uniform mesh of width h on $[a, b]$, and ψ_h is chosen to be the interpolant of ψ in this space. In that case, $\beta_j = \psi(a + jh)$.

One can obtain a simple error estimate for this approximation scheme by comparing the approximate solution to the elliptic projection $w_h(t) \in V_h^0$, satisfying

$$a(u(t) - w_h(t), v_h) = 0, \quad v_h \in V_h^0.$$

We showed previously that if V_h consists of piecewise polynomials of degree $\leq r$, and u is sufficiently smooth, then

$$\begin{aligned}\|u(t) - w_h(t)\|_E &\leq Ch^r \|u^{(r+1)}(t)\|_{L^2} \leq Ch^r \|u(t)\|_{r+1}, \\ \|u(t) - w_h(t)\| &\leq Ch^{r+1} \|u^{(r+1)}(t)\|_{L^2} \leq Ch^{r+1} \|u(t)\|_{r+1},\end{aligned}$$

where

$$\|u\|_r^2 = \sum_{k=0}^r \|u^{(k)}\|_{L^2}^2.$$

Using these results, we can establish the following result.

Theorem 36. *If V_h consists of piecewise polynomials of degree $\leq r$, the initial approximation ψ_h satisfies $\|\psi - \psi_h\|_{L^2} \leq Ch^{r+1} \|\psi\|_{r+1}$, and u is sufficiently smooth, then for $t \geq 0$,*

$$\|u(t) - u_h(t)\|_{L^2} \leq Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^t \|u_t\|_{r+1} ds \right].$$

Proof. We estimate the error by writing $u - u_h = (u - w_h) + (w_h - u_h)$. From the above, we have

$$\begin{aligned}\|u(t) - w_h(t)\| &\leq Ch^{r+1} \|u(t)\|_{r+1} \leq Ch^{r+1} \|u(0) + \int_0^t u_t(s) ds\|_{r+1} \\ &\leq Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1} ds \right].\end{aligned}$$

It thus remains to estimate $\|u_h - w_h\|$. Using the continuous and discrete variational formulations, we have

$$(\partial[u_h - u]/\partial t, v) + a(u_h - u, v) = 0, \quad v \in V_h^0.$$

Applying the above and the definition of $w_h(t)$, we get

$$\begin{aligned}(\partial[u_h - w_h]/\partial t, v) + a(u_h - w_h, v) &= (\partial[u_h - u]/\partial t, v) + a(u_h - u, v) \\ &\quad + (\partial[u - w_h]/\partial t, v) + a(u - w_h, v) = (\partial[u - w_h]/\partial t, v), \quad v \in V_h^0.\end{aligned}$$

Choosing $v = u_h - w_h$, and observing that

$$\|u_h - w_h\| \frac{d}{dt} \|u_h - w_h\| = \frac{1}{2} \frac{d}{dt} \|u_h - w_h\|^2 = ([u_h - w_h]_t, u_h - w_h),$$

we get

$$\|u_h - w_h\| \frac{d}{dt} \|u_h - w_h\| + \|u_h - w_h\|_E^2 = ([u - w_h]_t, u_h - w_h) \leq \| [u - w_h]_t \| \|u_h - w_h\|.$$

Hence,

$$\frac{d}{dt} \|u_h - w_h\| \leq \| [u - w_h]_t \|.$$

Integrating this equation between 0 and t , we get

$$\begin{aligned} \|u_h(t) - w_h(t)\| &\leq \|u_h(0) - w_h(0)\| + \int_0^t \|[u - w_h]_t(s)\| ds \\ &\leq \|u_h(0) - u(0)\| + \|u(0) - w_h(0)\| + \int_0^t \|[u - w_h]_t(s)\| ds. \\ &\leq \|\psi - \psi_h\| + Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1} ds \right]. \end{aligned}$$

Using the triangle inequality, and combining estimates, we then obtain

$$\|u(t) - u_h(t)\| \leq \|u(t) - w_h(t)\| + \|u_h(t) - w_h(t)\| \leq \|\psi - \psi_h\| + Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^t \|u_t(s)\|_{r+1} ds \right],$$

and the result follows immediately from the approximation assumption on ψ_h . \square

13.2. Fully discrete schemes. One way to get a fully discrete scheme is to combine the use of finite elements to discretize the spatial variable with a finite difference approximation in time. For example, if we approximate u_t by the backward Euler approximation, we get the scheme: Find $U^n \in V_h^0$, satisfying $U^0 = \phi_h$ and for $n \geq 0$

$$([U^{n+1} - U^n]/k, v) + a(U^{n+1}, v) = (f^{n+1}, v) \quad v \in V_h^0.$$

Using the matrices defined previously, and defining $U^n(x) = \sum_{j=1}^m \alpha^n \phi_j(x)$, the discrete variational formulation above corresponds to the linear system

$$(M + kA)\alpha^{n+1} = M\alpha^n + kF^{n+1}, \quad n = 0, 1, \dots$$

Another choice is the Crank-Nicholson-Galerkin method, which has the form: Find $U^n \in V_h^0$, satisfying $U^0 = \phi_h$ and for $n \geq 0$

$$([U^{n+1} - U^n]/k, v) + a([U^{n+1} + U^n]/2, v) = ([f^{n+1} + f^n]/2), \quad v \in V_h^0.$$

In this case, we get the linear system

$$(M + kA)\alpha^{n+1} = (M - kA)\alpha^n + k(F^{n+1} + F^n)/2, \quad n = 0, 1, \dots$$

For the backward Euler method, we have the following error estimate ($t_n = nk$).

Theorem 37. *Under the assumptions of the previous theorem, we have*

$$\|u(t_n) - U^n\| \leq Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} ds \right] + k \int_0^{t_n} \|u_{tt}(s)\| ds, \quad n \geq 0.$$

Proof. As before, we write $u(t_n) - U^n = (u(t_n) - W^n) + (W^n - U^n)$, where $W^n = w_h(t_n)$. From our previous result, we have

$$\|u(t_n) - W^n\| \leq Ch^{r+1} \left[\|\psi\|_{r+1} + \int_0^{t_n} \|u_t(s)\|_{r+1} ds \right].$$

To estimate $U^n - W^n$, we again use our continuous and discrete variational formulations to first obtain:

$$\begin{aligned} ([(U - u)^{n+1} - (U - u)^n] / k, v) + a((U - u)^{n+1}, v) &= (f^{n+1}, v) - ([u^{n+1} - u^n] / k, v) - a(u^{n+1}, v) \\ &= (u_t^{n+1} - [u^{n+1} - u^n] / k, v), \quad v \in V_h^0. \end{aligned}$$

Then, using the definition of W^n , we get

$$\begin{aligned} ([(U - W)^{n+1} - (U - W)^n] / k, v) + a((U - W)^{n+1}, v) &= ([(U - u)^{n+1} - (U - u)^n] / k, v) \\ &\quad + a((U - u)^{n+1}, v) + ([(u - W)^{n+1} - (u - W)^n] / k, v) + a((u - W)^{n+1}, v) \\ &= (u_t^{n+1} - [u^{n+1} - u^n] / k, v) + ([(u - W)^{n+1} - (u - W)^n] / k, v) \equiv (\rho^n, v) \quad v \in V_h^0. \end{aligned}$$

Choosing $v = (U - W)^{n+1}$, we get

$$\begin{aligned} \|(U - W)^{n+1}\|^2 + k\|(U - W)^{n+1}\|_E^2 &= ((U - W)^n, (U - W)^{n+1}) + k(\rho^n, (U - W)^{n+1}) \\ &\leq [\|(U - W)^n\| + k\|\rho^n\|] \|(U - W)^{n+1}\|. \end{aligned}$$

Hence,

$$\|(U - W)^{n+1}\| \leq \|(U - W)^n\| + k\|\rho^n\|.$$

Iterating this equation, we get

$$\|(U - W)^n\| \leq \|(U - W)^0\| + k \sum_{j=0}^{n-1} \|\rho^j\|.$$

Using integration by parts, we note that

$$\begin{aligned} k^{-1} \int_{t_j}^{t_{j+1}} [(s - t_j)u_{tt}(s) + (u - w_h)_t(s)] ds \\ &= k^{-1} [(s - t_j)u_t(s)]_{s=t_j}^{s=t_{j+1}} + k^{-1} \int_{t_j}^{t_{j+1}} [-u_t(s) + (u - w_h)_t(s)] ds \\ &= u_t^{j+1} - [u^{j+1} - u^j] / k + [(u - W)^{j+1} - (u - W)^j] / k = \rho^j. \end{aligned}$$

Hence,

$$\begin{aligned} \|\rho^j\| &\leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + \|(u - w_h)_t(s)\|] ds \\ &\leq k^{-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] ds. \end{aligned}$$

Finally,

$$\begin{aligned} k \sum_{j=0}^{n-1} \|\rho^j\| &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [k\|u_{tt}(s)\| + Ch^{r+1}\|u_t(s)\|_{r+1}] ds \\ &\leq k \int_{t_0}^{t_n} \|u_{tt}(s)\| ds + Ch^{r+1} \int_{t_0}^{t_n} \|u_t(s)\|_{r+1} ds. \end{aligned}$$

The theorem follows by combining all these results. \square