

13.12. **Discontinuous Galerkin methods for ordinary differential equations.** Reference: Delfour, Hager and Trochu, Math. Comp. (36) 1981, pp. 455-472.

Consider the problem $y' = f(t, y)$, $y(0) = \alpha$. Let $I_m = (t_{m-1}, t_m)$ and

$$V_h = \{v : v|_{I_m} \in P_k(I_m), m = 1, \dots, N\}.$$

The approximation scheme is: Find $y_h \in V_h$ and $y_h(0^-)$ and $y_h(T^+)$ such that

$$\int_{t_{j-1}}^{t_j} [dy_h/dt - f(t, y_h)]v dt + v(t_{j-1}^+) \alpha_{j-1} [y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] \\ + v(t_j^-) (1 - \alpha_j) [y_h(t_j^+) - y_h(t_j^-)] = 0, \quad \forall v \in P_k(I_m),$$

where $\alpha_0, \dots, \alpha_N$ are scalars with $0 \leq \alpha_j \leq 1$ and we evaluate the integrals using a $k + 1$ point quadrature formula exact for polynomials of degree $\leq 2k + 1$.

In this method, we produce a discontinuous piecewise polynomial approximation to the solution, i.e., the value $y_h(t_j^-)$ gives an approximation to $y(t_j)$ inside the interval (t_{j-1}, t_j) and $y_h(t_j^+)$ gives an approximation to $y(t_j)$ inside the interval (t_j, t_{j+1}) . Note that the true solution of the initial value problem satisfies the equation given by the approximation scheme, since it satisfies the differential equation and for all j , the jump terms $y_h(t_j^+) - y_h(t_j^-) = 0$.

We will assume that either (i) $\alpha_0 = 0$ and $\alpha_j \neq 1, j = 1, \dots, N$, or (ii) $\alpha_N = 1$ and $\alpha_j \neq 0, j = 0, \dots, N - 1$. In case (i), the unknown $y(0^-)$ disappears and in case (ii), the unknown $y(T^+)$ disappears. Hence we have $N(k + 1) + 1$ unknowns and $N(k + 1) + 1$ equations, where one of the unknowns and equations is determined by the initial condition and the rest from the above equation.

Consider some special cases: $k = 0$. Then y_h is a constant on each subinterval so $y_h(t_j^-) = y_h(t_{j-1}^+)$. Taking $v = 1$, and applying the midpoint quadrature rule (a one-point formula exact for linear polynomials), we get:

$$y_h(t_j^-) - y_h(t_{j-1}^+) - hf(t_{j-1/2}, y_h(t_{j-1/2})) + \alpha_{j-1} [y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] + (1 - \alpha_{j-1}) [y_h(t_j^+) - y_h(t_j^-)] = 0.$$

Now suppose $\alpha_j = 0$ for all j . Then

$$y_h(t_j^+) - y_h(t_{j-1}^+) = hf(t_{j-1/2}, y_h(t_{j-1}^+)), \quad j = 1, \dots, N.$$

In this case, $y_h(t_j^+)$ denotes the approximation to the true solution $y(t_j)$, so the method is similar to explicit Euler, except that t is evaluated at $t_{j-1/2}$ instead of t_{j-1} . In this case the starting value is $y_h(t_0^+)$ and $y_h(t_0^-)$ is not part of the method.

Next suppose $\alpha_j = 1$ for all j ($k = 0$). Then

$$y_h(t_j^-) - y_h(t_{j-1}^-) = hf(t_{j-1/2}, y_h(t_j^-)), \quad j = 1, \dots, N.$$

In this case, $y_h(t_j^-)$ denotes the approximation to the true solution $y(t_j)$, so the method is similar to implicit Euler, except that t evaluated at $t_{j-1/2}$ instead of t_j . In this case the starting value is $y_h(t_0^-)$ and $y_h(T^+)$ is not part of the method.

The methods with the choice $\alpha_j = 1$ are equivalent to implicit Runge-Kutta methods.

When $k = 1$, y_h will be a linear polynomial on the subinterval (t_{j-1}, t_j) which we write as:

$$y_h(t) = [1 - (t - t_{j-1})/h]y_h(t_{j-1}^+) + [(t - t_{j-1})/h]y_h(t_j^-).$$

If $\alpha_j = 1$, the equations are

$$\int_{t_{j-1}}^{t_j} [dy_h/dt - f(t, y_h)]v dt + v(t_{j-1}^+)[y_h(t_{j-1}^+) - y_h(t_{j-1}^-)] = 0,$$

where we evaluate the integral using a 2-point Gauss formula. We now have two unknowns $y_h(t_{j-1}^+)$ and $y_h(t_j^-)$, and get two equations by choosing $v = 1$ and $v = t$. We thus need to solve a 2×2 system of non-linear equations.