Math 477 / Quiz 5 (Solution)

1. The joint density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y}{3} & \text{if } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

Sln: No, they are not. It is impossible to write $f_{X,Y}(\cdot,\cdot)$ as the product of two functions, for all $x, y \in \mathbb{R}$, one depending on x only and the other one depending on y only.

(b) Find the density function of Y.

Sln: We showed in class that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Hence, for $0 < y < 2$,

$$f_Y(y) = \frac{1}{3}\int_0^1 (x+y)dx = \frac{1}{3}(1/2 + y)$$

Thus,

$$f_Y(y) = \begin{cases} \frac{1}{3}(1/2+y) & \text{if } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find $\mathbb{P}(X + Y > 1)$

Sln: I'm sorry for not drawing the region here, but I'm gonna leave that to you guys. Please draw on a cartesian plane the rectangle $[0,1] \times [0,2]$ where $f_{X,Y}(\cdot,\cdot)$ is strictly positive. Draw the line $y = 1-x$. To calculate the probability, we want to integrate over the region inside the rectangle $[0,1]\times[0,2]$ above the line $y = 1-x$. That's a trapezoid we can describe as $\{(x,y) | 0 < x < 1, 1-x < y < 2\}$. Therefore,

$$\begin{aligned} \mathbb{P}(X+Y>1) &= \int_0^1 \int_{1-x}^2 \frac{x+y}{3} dy dx \\ &= \frac{1}{3}\int_0^1 \left(x^2/2 + 2x + 3/2\right) dx \\ &= \frac{1}{3}(1/6 + 1 + 3/2) \\ &= 8/9 \end{aligned}$$