Math 421 / Quiz 2 (Solution)

1. Use the Laplace transform to solve the given IVP

$$\begin{cases} y'' - 8y' + 20y = te^t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Useful formula:

$$\frac{1}{(s-1)^2(s^2-8s+20)} = \frac{\frac{6}{169}}{s-1} + \frac{\frac{1}{13}}{(s-1)^2} + \frac{\frac{-6}{169}s + \frac{29}{169}}{s^2-8s+20}$$

Solution: We start taking the Laplace transform of the differential equation to get

$$s^2Y(s) - 8sY(s) + 20Y(s) = \frac{1}{(s-1)^2}$$

with the help of our formula table. Hence

$$Y(s) = \frac{1}{(s-1)^2(s^2-8s+20)}$$

Now notice that $s^2 - 8s + 20$ can not be factorized anymore and so we proceed to decompose that fraction into partial fractions. You know how to do that... you should get

$$Y(s) = \frac{\frac{6}{169}}{s-1} + \frac{\frac{1}{13}}{(s-1)^2} + \frac{\frac{-6}{169}s + \frac{29}{169}}{s^2-8s+20}$$

We are ready then to take the inverse Laplace tranform of $Y(s)$ again using our formula table. Let's rewrite $Y(s)$ first:

$$Y(s) = \frac{\frac{6}{169}}{s-1} + \frac{\frac{1}{13}}{(s-1)^2} + \frac{\frac{-6}{169}(s-4) + \frac{5}{169}}{(s-4)^2+4}$$

which is the same as

$$Y(s) = \frac{\frac{6}{169}}{s-1} + \frac{\frac{1}{13}}{(s-1)^2} + \frac{\frac{-6}{169}(s-4)}{(s-4)^2+4} + \frac{\frac{5}{169}}{(s-4)^2+4}$$

Thus,

$$y(t) = \frac{6}{169}e^t + \frac{1}{13}te^t - \frac{6}{169}e^{4t}\cos(2t) + \frac{5}{338}e^{4t}\sin(2t)$$