Math 421: Advanced Calculus for Engineers

Final Exam (Solution)

July 19th, 2007

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1. Answer true/false. Justify your answer.

   (a) **(5 pts)** The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ has 4 linearly independent eigenvectors.

   Sln: TRUE. The matrix $\mathbf{A}$ is symmetric. Thus it has 4 orthogonal eigenvectors, which in particular are linearly independent.

   (b) **(5 pts)** The functions $f(x) = x^2 - 1$ and $g(x) = x^5$ are orthogonal on the interval $[-1, 1]$.

   Sln: TRUE, since $(f, g) = \int_{-1}^{1} (x^2 - 1)x^5 dx = 0$.

2. **(25 pts)** Solve the initial value problem

$$y'' - y = \delta(t-2)$$

with $y(0) = 3$, $y'(0) = 4$.

Sln: Taking the Laplace transform we get

$$s^2 Y(s) - 3s - 4 - Y(s) = e^{-2s}$$

where $Y(s)$ is the Laplace transform of y. Thus, solving for $Y(s)$,

$$Y(s) = \frac{e^{-2s}}{s^2-1} + \frac{3s+4}{s^2-1}$$

Now, expanding in partial fractions we get

$$Y(s) = e^{-2s}\left(\frac{1/2}{s-1} + \frac{-1/2}{s+1}\right) + \left(\frac{7/2}{s-1} + \frac{-1/2}{s+1}\right)$$

and we are ready to take the inverse Laplace transform. Using the second translation theorem for the first term we conclude that

$$y(t) = \left(\frac{1}{2}e^{t-2} - \frac{1}{2}e^{-(t-2)}\right)\mathcal{U}(t-2) + \frac{7}{2}e^{t} - \frac{1}{2}e^{-2t}$$

where $\mathcal{U}(t-2)$ is the Heaviside function with jump at 2.

3. Let $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(a) **(10 pts)** Compute the characteristic polynomial of $\mathbf{A}$.

Sln: $p(\lambda) = (\lambda - 1)^2(\lambda + 1)^2$

(b) **(20 pts)** $\lambda = 1$ and $\lambda = -1$ are the only eigenvalues of $\mathbf{A}$. Find the corresponding eigenvectors and decide whether $\mathbf{A}$ is diagonalizable or not.

Sln: For $\lambda = 1$ one finds that there is only one eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (modulo multiplication by scalars), and for $\lambda = -1$ one finds that there is only one eigenvector $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (modulo multiplication by scalars). Therefore, there are only two linearly independet eigenvectors and so the matrix A is not diagonalizable.

(c) **(10 pts)** What is the rank of **A**? Why?

Sln: Since $det(A) \neq 0$ (because 0 is not an eigenvalue), the matrix A is invertible and thus it has have rank 4.

(d) **(10 pts)** How many solutions does the linear system $\mathbf{A}x = 0$ have? Why?

Sln: Exactly one, the trivial solution. This follows since A is invertible (or since the rank of A is 4).

4. **(25 pts)** Consider the function $f(x) = |x|$. Find the full Fourier series of f as explicitly as you can in the interval $[-\pi, \pi]$.

Sn: Note that f is even, so there is no need of computing the b_n. Only the a_n need to be computed:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi} \int_{0}^{\pi} xdx = \pi$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx)dx = \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}$$

for $n \geq 1$.

5. Let $f(x) = \pi x^2 - 2x^3$ on $[0, \pi]$, and $g(x)$ be the sum of the whole Fourier **sine** series for f, and $h(x)$ be the sum of the whole Fourier **cosine** series for f. Compute the following numbers:
Hint: You don't need (and please don't do it) to find all those Fourier series to answer this. RELAX and THINK.

Sln: First of all I got relaxed and started thinking. Got it. g and h are the sums of the sine and cosine Fourier series and I know what the Fourier series are equal to by Dirichlet's Theorem. So $g(x)$ will be equal to the periodic extension of f when I define it over $[-\pi, 0]$ as an odd function wherever that function is continuous and equal to the average wherever there's a jump. Similarly with h but extending f as an even function first over $[-\pi, 0]$ and then periodically. DRAW GRAPHS and my calculations below will be easy to follow.

(a) **(5 pts)** $f(\pi)$, $g(\pi)$, $h(\pi)$

Sln: $f(\pi)$ is simply $f(\pi) = \pi^3 - 2\pi^3 = -\pi^3$. The extension of f as an odd function jumps at π, so $g(\pi) = \frac{-\pi^3+\pi^3}{2} = 0$. the extension of f as an even function is continuous everywhere, so $h(\pi) = -\pi^3$.

(b) **(5 pts)** $f(1)$, $g(1)$, $h(1)$

Sln: 1 is in the interval $[0, \pi]$ where f is continuous so $f(1) = g(1) = h(1) = -\pi^3$.

(c) **(5 pts)** $g(-\pi/3)$, $h(-\pi/3)$

Sln: Since g is equal to the odd extension of f and f doesn't jump at $\pi/3$, we have that $g(-\pi/3) = -f(\pi/3) = -\frac{\pi^3}{27}$. Since h is equal to the even extension of f and f doesn't jump at $\pi/3$, we also have that $h(-\pi/3) = \pi^3/27$.

6. In this problem, separation of variables will be used to analyze the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \qquad (*)$$

(a) **(10 pts)** If $u(x,t) = X(x)T(t)$, show that $X(x)$ and $T(t)$ must satisfy that

$$\frac{T'(t)}{T(t)} - 1 = \frac{X''(x)}{X(x)}$$

for $u(x,t)$ to be a solution of (*).

Sln: Let $u(x,t) = X(x)T(t)$. Hence $\frac{\partial u}{\partial t} = X(x)T'(t)$ and $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$. Thus, if u solves (*) then,

$$X(x)T'(t) = X''(x)T(t) + X(x)T(t)$$

which is equivalent to

$$\frac{T'(t)}{T(t)} - 1 = \frac{X''(x)}{X(x)}$$

after dividing by $X(x)T(t)$ and manipulating the equation algebraically.

(b) **(5 pts)** Deduce a couple of ordinary differential equations that $X(x)$ and $T(t)$ must satisfy, respectively.

Sln: The equation above holds for all x and t, arbitrarily, but it depends on x only on the right hand side and depends on t only on the left hand side. Thus they both have to be equal to a constant which I will cal $= \lambda$. Therefore,

$$\frac{X''(x)}{X(x)} = -\lambda = \frac{T'(t)}{T(t)} - 1$$

which implies that

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T'(t) + (\lambda - 1)T(t) &= 0 \end{aligned}$$

(c) **(10 pts)** Suppose the solution $u(x,t)$ found in (a)-(b) also satisfies the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$ for all t. How are $X(x)$ and $T(t)$ further restricted?

Sln: $u(0,t) = 0$ and $u(\pi,t) = 0$ for all t implies that for finding non trivial solutions to (*) we should impose $X(0) = 0$ and $X(\pi) = 0$. That is, $X(x)$ must solve the Sturm-Liouville problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

We know that the non trivial solutions to this problem are $X_n(x) = \sin(nx)$ for $n \geq 1$ and $\lambda_n = n^2$. Now, T_n must satify then

$$T_n'(t) + (n^2 - 1)T_n(t) = 0$$

whose solution is $T_n(t) = e^{-(n^2-1)}$ (modulo a constant).

(d) **(20 extra points)** Use your answer to (a)-(c) to write a formula (it will be an infinite series) for the most general solution to (*) which satisfies the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$.

Sln: So far we know that $u_n(x,t) := X_n(x)T_n(t) = \sin(nx)e^{-(n^2-1)t}$ are solutions to

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \\ u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$$

and since this is a homogenoues PDE with homogenoues boundary conditions, a linear combination of solutions will give us another solution. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-(n^2-1)t}$$

is the general solution we were looking for.

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