Math 421: Advanced Calculus for Engineers

Final Exam

July 19th, 2007

Name:

1. Answer true/false. Justify your answer.

   (a) **(5 pts)** The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ has 4 linearly independent eigenvectors.

   (b) **(5 pts)** The functions $f(x) = x^2 - 1$ and $g(x) = x^5$ are orthogonal on the interval $[-1, 1]$.

2. **(25 pts)** Solve the initial value problem

$$y'' - y = \delta(t-2)$$

with $y(0) = 3$, $y'(0) = 4$.

3. Let $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(a) **(10 pts)** Compute the characteristic polynomial of $\mathbf{A}$.

(b) **(20 pts)** $\lambda = 1$ and $\lambda = -1$ are the only eigenvalues of $\mathbf{A}$. Find the corresponding eigenvectors and decide whether $\mathbf{A}$ is diagonalizable or not.

(c) **(10 pts)** What is the rank of $\mathbf{A}$? Why?

(d) **(10 pts)** How many solutions does the linear system $\mathbf{A}x = 0$ have? Why?

4. **(25 pts)** Consider the function $f(x) = |x|$. Find the full Fourier series of f as explicitly as you can in the interval $[-\pi, \pi]$.

5. Let $f(x) = \pi x^2 - 2x^3$ on $[0, \pi]$, and $g(x)$ be the sum of the whole Fourier **sine** series for f, and $h(x)$ be the sum of the whole Fourier **cosine** series for f. Compute the following numbers:

   (a) **(5 pts)** $f(\pi)$, $g(\pi)$, $h(\pi)$

   (b) **(5 pts)** $f(1)$, $g(1)$, $h(1)$

   (c) **(5 pts)** $g(-\pi/3)$, $h(-\pi/3)$

   Hint: You don't need (and please don't do it) to find all those Fourier series to answer this. RELAX and THINK.

6. In this problem, separation of variables will be used to analyze the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \qquad (*)$$

(a) **(10 pts)** If $u(x,t) = X(x)T(t)$, show that $X(x)$ and $T(t)$ must satisfy that

$$\frac{T'(t)}{T(t)} - 1 = \frac{X''(x)}{X(x)}$$

for $u(x,t)$ to be a solution of (*).

(b) **(5 pts)** Deduce a couple of ordinary differential equations that $X(x)$ and $T(t)$ must satisfy, respectively.

(c) **(10 pts)** Suppose the solution $u(x,t)$ found in (a)-(b) also satisfies the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$ for all t. How are $X(x)$ and $T(t)$ further restricted?

(d) **(20 extra points)** Use your answer to (b) to write a formula (it will be an infinite series) for the most general solution to (*) which satisfies the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$.

Math 421 / Formula Sheet

Function	Laplace transform
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$, where $F(s)=\mathcal{L}\{f(t)\}$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$, where $F(s)=\mathcal{L}\{f(t)\}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$, where $F(s)=\mathcal{L}\{f(t)\}$
$\delta(t-t_0)$	$e^{-t_0 s}$
$f * g$	$F(s)G(s)$, where $F(s)=\mathcal{L}\{f(t)\}$ and $G(s)=\mathcal{L}\{g(t)\}$
f'	$sF(s)-f(0)$, where $F(s)=\mathcal{L}\{f(t)\}$
f	$\int_0^\infty e^{-st}f(t)dt$

Some trig identities

$$\sin a\cos b = \tfrac{1}{2}\left(\sin(a+b)+\sin(a-b)\right)$$
$$\cos a\cos b = \tfrac{1}{2}\left(\cos(a+b)+\cos(a-b)\right)$$
$$\sin a\sin b = \tfrac{1}{2}\left(\cos(a+b)-\cos(a-b)\right)$$
$$\sin^2 x = 1 - cos^2 x = \tfrac{1-\cos(2x)}{2}$$

Let me remind you that that the coefficients of the Fourier series of f on $[-p,p]$

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{p}x\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{p}x\right)$$

are given by

$$a_n=\frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx,\quad b_n=\frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx$$

The solutions to the Sturm-Liouville problem

$$\begin{cases} X''+\lambda X=0 \\ X(0)=0 \\ X(L)=0 \end{cases}$$

are given by $X_n(x)=\sin\left(\frac{n\pi}{L}x\right)$ with $\lambda_n=\frac{n^2\pi^2}{L^2}$ for $n\geq 1$.

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