Math 421: Advanced Calculus for Engineers

Second Exam (Solution)

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1. Answer true/false. Justify your answer.

   (a) **(5 pts)** If $\mathbf{A} = 5\mathbf{B}$, then $\det \mathbf{A} = 5 \det \mathbf{B}$.
   Sln: FALSE. Consider $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that $\det \mathbf{A} = 25$ but $5 \det \mathbf{B} = 5$.

   (b) **(5 pts)** If 0 is an eigenvalue of $\mathbf{A}$, then $\mathbf{A}$ is singular.
   Sln: TRUE. If 0 is an eigenvalue of $\mathbf{A}$, then $\det(\mathbf{A} - 0\mathbf{I}) = 0$, that is $\det \mathbf{A} = 0$. This implies that $\mathbf{A}$ is singular.

   (c) **(5 pts)** The matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ cannot be diagonalized.
   Sln: TRUE. If it were possible to diagonalize $\mathbf{A}$, then there would exist a diagonal matrix $\mathbf{D}$ (with the eigenvalues of $\mathbf{A}$ along the diagonal) and an invertible matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1} A \mathbf{P}$. But the eigenvalues of $\mathbf{A}$ are all equal to 0 ($\mathbf{A}$ is a triangular matrix and the diagonal has only zeros), hence $\mathbf{D}$ would be the zero matrix. That is, $0 = \mathbf{P}^{-1} A \mathbf{P}$. Multiplying by $\mathbf{P}$ from the left and by $\mathbf{P}^{-1}$ from the right, we would get that $\mathbf{A}$ was the zero matrix, which is not the case. Therefore, $\mathbf{A}$ cannot be diagonalized.

   (d) **(5 pts)** The product of two symmetric matrices is symmetric.
   Sln: FALSE. Consider $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
   Both $\mathbf{A}$ and $\mathbf{B}$ are symmetric, but $\mathbf{AB} = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ is not.

2. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. $\mathbf{A}$ has three different eigenvalues and those are 0, 1 and 2.

   (a) **(10 pts)** Find a basis of $\mathbf{R}^3$ consisting of eigenvectors of $\mathbf{A}$.
   Sln: Since the three eigenvalues are different, the set of corresponding eigenvectors will consist of three linearly independet eigenvector in $\mathbf{R}^3$. So they will be a basis for $\mathbf{R}^3$. Let's find them:

      i. $\lambda = 0$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

      ii. $\lambda = 1$.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

      iii. $\lambda = 2$.

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

   So $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a base of $\mathbf{R}^3$ consisting of eigenvectors of $\mathbf{A}$.

   (b) **(10 pts)** Find a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.
   Sln: We know that such a matrix $\mathbf{D}$ has the eigenvalues of $\mathbf{A}$ along the diagonal and that $\mathbf{P}$ has the eigenvectors of $\mathbf{A}$, corresponding to the eigenvalues written on D with the same order, as its columns. That is,

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(c) **(10 pts)** Find $\mathbf{P}^{-1}$ by the method of cofactors.
Sln: Let's find the determinant of $\mathbf{P}$ first:

$$\det \mathbf{P} = (-1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -2$$

Now, the matrix of cofactors of $\mathbf{A}$:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & -1 \\ -1 & 0 & -1 \end{pmatrix}$$

Thus,

$$\mathbf{P^{-1}} = \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

(d) **(10 pts)** Compute $\mathbf{A}^6$.
Sln: From $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ it follows that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and hence that $\mathbf{A}^6 = \mathbf{P}\mathbf{D}^6\mathbf{P}^{-1}$. So we just need to multiply these three matrices to get $\mathbf{A}^6$:

$$\begin{aligned}
\mathbf{A}^6 &= \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^6 \end{pmatrix} \mathbf{P}^{-1} \\
&= \begin{pmatrix} 0 & -1 & 2^6 \\ 0 & 1 & 0 \\ 0 & 0 & 2^6 \end{pmatrix} \mathbf{P}^{-1} \\
&= \begin{pmatrix} 0 & -1 & 2^6 \\ 0 & 1 & 0 \\ 0 & 0 & 2^6 \end{pmatrix} \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} \\
\mathbf{A}^6 &= \begin{pmatrix} 2^5 & -1+2^5 & 2^5 \\ 0 & 1 & 0 \\ 2^5 & 2^5 & 2^5 \end{pmatrix} = \begin{pmatrix} 32 & 31 & 32 \\ 0 & 1 & 0 \\ 32 & 32 & 32 \end{pmatrix}
\end{aligned}$$

3. **(15 pts)** Let $\mathbf{M} = \begin{pmatrix} a & 0 & 0 & a & a \\ 0 & 0 & b & 0 & 0 \\ c & c & 0 & c & 0 \\ d & 0 & d & d & 0 \\ 0 & e & 0 & e & e \end{pmatrix}$. Compute its determinant.

Sln: There are several ways of computing this. None of them are particularly harder than the other one... just a preference. Here is one: I start factoring a, b, c, d and e to get

$$\det \mathbf{M} = abcde \begin{vmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$$

The second row has only one non-zero entry so

$$\det \mathbf{M} = -abcde \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

Now, $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$:

$$\det \mathbf{M} = -abcde \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

and $R_4 = R_4 - R_2$ leave us with

$$\det \mathbf{M} = -abcde \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = abcde \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

Hence $\det \mathbf{M} = -abcde$.

4. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

(a) **(15 pts)** Compute $\mathbf{A}^{-1}$ by row operations.
Sln: Augment the matrix $\mathbf{A}$ with the identity matrix and do $R_2 = R_2 - 2R_1$ to get

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply the second row by -1 and do $R_3 = R_3 - R_2$ to get

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right)$$

After $R_1 = R_1 - 2R_2$ and dividing by -4 the third row one gets...

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -9 & -3 & 2 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/4 & -1/4 \end{array}\right)$$

Almost done. Now we need to do $R_1 = R_1 + 9R_3$ and $R_2 = R_2 - 6R_3$ to get the identity matrix to the left of the line:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -9 & -3 & 2 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/4 & -1/4 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1/4 & -9/4 \\ 0 & 1 & 0 & -1 & 1/2 & 3/2 \\ 0 & 0 & 1 & 1/2 & -1/4 & -1/4 \end{array}\right)$$

Therefore, $\mathbf{A}^{-1} = \begin{pmatrix} 3/2 & -1/4 & -9/4 \\ -1 & 1/2 & 3/2 \\ 1/2 & -1/4 & -1/4 \end{pmatrix}$

(b) **(10 pts)** Solve the linear system $\mathbf{AX} = \mathbf{B}$ where $\mathbf{B} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Sln:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 3/2 & -1/4 & -9/4 \\ -1 & 1/2 & 3/2 \\ 1/2 & -1/4 & -1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

5. **(Extra problem, 5 pts)** Calculate the determinant of $adj(\mathbf{A})$, the adjoint of $\mathbf{A}$, in terms of $\det \mathbf{A}$.
Sln: I should have added in the statement of the problem that $A \in M_{n\times n}$ and that $\mathbf{A}$ is nonsingular. We know that

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} adj(\mathbf{A})$$

Thus

$$adj(\mathbf{A}) = (\det \mathbf{A})\mathbf{A}^{-1}$$

and when we calculate the determinant we get

$$\det(adj(\mathbf{A})) = (\det \mathbf{A})^n \det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{n-1}$$

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