Math 421: Advanced Calculus for Engineers

First Exam (Solution)

June 14th, 2007

Name: Eduardo Osorio

1. Answer true/false. Justify your answer.

   (a) **(5 pts)** The function $(e^t)^{10000}$ is not of exponential order.
   Sln: FALSE. $(e^t)^{10000} = e^{10000t}$, so this function IS of exponential order.

   (b) **(5 pts)** $F(s) = \frac{s^2}{s^2+1}$ is not the Laplace transform of a function that is piecewise continuous and of exponential order.
   Sln: TRUE. Since $\lim_{s\to\infty} F(s) = 1 \neq 0$.

   (c) **(5 pts)** If $\mathcal{L}\{f\} = F(s)$ and $\mathcal{L}\{g\} = G(s)$, then $\mathcal{L}\{fg\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.
   Sln: FALSE. For example, given that $\mathcal{L}\{1\} = \frac{1}{s}$, it is not true that $\mathcal{L}\{1 \cdot 1\} = \mathcal{L}\{1\}\mathcal{L}\{1\}$

2. Calculate the following Laplace transforms and inverse Laplace transforms:

   (a) **(7.5 pts)** $\mathcal{L}^{-1}\left\{\frac{1}{(2s-1)^4}\right\}$
   Sln:

$$\mathcal{L}^{-1}\left\{\frac{1}{(2s-1)^4}\right\} = \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{(s-1/2)^4}\right\} = \frac{1}{16\cdot 6}t^3e^{t/2}$$

   (b) **(7.5 pts)** $\mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\}$
   Sln:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}e^{-s}\right\} + \mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}e^{-s}\right\} \\ &= \cos{(t-\pi)}\mathcal{U}(t-\pi) + \sin{(t-\pi)}\mathcal{U}(t-\pi) \\ &= -(\cos t + \sin t)\mathcal{U}(t-\pi)\end{aligned}$$

   (c) **(5 pts)** $\mathcal{L}\{te^{8t}\}$
   Sln:

$$\mathcal{L}\left\{te^{8t}\right\} = \mathcal{L}\{t\}(s-8) = \frac{1}{(s-8)^2}$$

3. **(25 pts)** Use the Laplace transform to solve the linear system:

$$\begin{aligned} \frac{dx}{dt} + y &= t \\ 4x + \frac{dy}{dt} &= 0 \end{aligned}$$

such that $x(0) = 1$ and $y(0) = 2$.
Sln: Let's take the Laplace transform of both equations

$$\begin{aligned} sX(s) - x(0) + Y(s) &= \frac{1}{s^2} \quad (1) \\ 4X(s) + sY(s) - y(0) &= 0 \quad (2) \end{aligned}$$

and use our initial conditions to get the algebraic problem

$$\begin{aligned} sX(s) + Y(s) &= \frac{s^2+1}{s^2} \quad (3) \\ 4X(s) + sY(s) &= 2 \quad (4) \end{aligned}$$

After solving this system of equations (3) and (4) we end up with

$$\begin{aligned} X(s) &= \frac{s^2 - 2s + 1}{s(s^2-4)} \quad (5) \\ Y(s) &= \frac{2s^3 - 4s^2 + 4s - 4}{s^2(s^2-4)} \quad (6) \end{aligned}$$

Now, decomposing into partial fractions, we get

$$\begin{aligned} X(s) &= \frac{-1/4}{s} + \frac{1/8}{s-2} + \frac{9/8}{s+2} \quad (7) \\ Y(s) &= \frac{1}{s^2} - \frac{1/4}{s-2} + \frac{9/4}{s+2} \quad (8) \end{aligned}$$

and after taking the inverse Laplace transform we get the solution to our initial system of equations

$$\begin{aligned} x(t) &= -\frac{1}{4} + \frac{1}{8}e^{2t} + \frac{9}{8}e^{-2t} \quad (9) \\ y(t) &= t - \frac{1}{4}e^{2t} + \frac{9}{4}e^{-2t} \quad (10) \end{aligned}$$

4. **(20 pts)** Let $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$. Calculate the quantities that make sense:

(a) $\mathbf{AB}$

Sln: $\mathbf{AB} = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}$

(b) $\mathbf{BA}$

Sln: $\mathbf{BA} = 4 + 10 + 18 = 32$

(c) $\mathbf{A^T B^T}$

Sln: $\mathbf{A^T B^T}$ is the transpose of $\mathbf{BA}$, hence $\mathbf{A^T B^T} = 32$

(d) $\mathbf{B^T A^T}$

Sln: $\mathbf{B^T A^T}$ is the transpose of $\mathbf{AB}$, hence $\mathbf{B^T A^T} = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}$

(e) $((\mathbf{A} + \mathbf{B^T})^\mathbf{T} - \mathbf{A^T})^\mathbf{T} - \mathbf{B^T}$

Sln: Too many $\mathbf{T}$'s perhaps? Simplify before calculating... you are gonna get $\mathbf{0}$.

5. (a) **(10 pts)** Use either Gaussian elimination or Gaussian-Jordan elimination to solve the given system or show that no solution exists

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 + x_3 &= 1 \end{aligned}$$

Sln: Consider the augmented matrix

$$\begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 3 & -1 & 2 & | & 5 \\ 2 & 1 & 1 & | & 1 \end{pmatrix}$$

I'll do Gaussian elimination. First, $R_2 = R_2 - 3R_1$ and $R_3 = R_3 - 2R_1$:

$$\begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 5 & -1 & | & -1 \\ 0 & 5 & -1 & | & -3 \end{pmatrix}$$

and now $R_3 = R_3 - R_2$:

$$\begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 5 & -1 & | & -1 \\ 0 & 0 & 0 & | & -2 \end{pmatrix}$$

Notice the last row doesn't have only zeroes and so the linear system has NO solutions.

(b) **(10 pts)** Use your solution to part (a) to get the number of solutions of the following linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Sln: Since this is a homogeneous system, after doing exactly the same row operations as in (a) we would end up with almost the same augmented matrix... just with zeroes to the right of the dotted line. Thus, we may see x_3 as a parameter and so the linear system will have infinite solutions.

6. **(Extra problem, 5 pts)** Calculate

$$\int_0^\infty te^{-t}\sin(2t)dt$$

Sln: Just notice that the integral is nothing but $\mathcal{L}\{t\sin(2t)\}$ evaluated at $s = 1$. But

$$\mathcal{L}\{t\sin(2t)\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

Therefore,

$$\int_0^\infty te^{-t}\sin(2t)dt = \frac{4}{25}$$

Eduardo Osorio