Name:______________________________

640:244–02 SPRING 2007 Quiz 2 January 31, 2007

1. Find an explicit solution of the initial value problem $y' = (2x+1)y^2$, $y(0) = 1/2$. What is the largest interval $\alpha < x < \beta$ (containing the initial point $x = 0$) in which your solution exists?

$$\frac{dy}{dx} = (2x+1)y^2 \Rightarrow \frac{dy}{y^2} = (2x+1)dx \overset{\int}{\Rightarrow} -\frac{1}{y} = x^2 + x + c$$

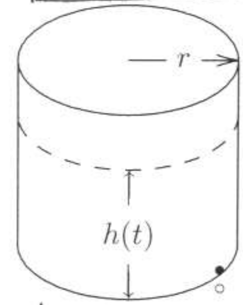
Since $y(0) = \frac{1}{2}$, $c = -2$. Thus, $y(x) = \dfrac{-1}{x^2+x-2}$

Let's find where y is well-defined, by finding where it is not:

$$x^2 + x - 2 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1-4(-2)}}{2} = \frac{-1 \pm 3}{2} \Leftrightarrow x = \begin{cases} 1 \text{ or} \\ -2 \end{cases}$$

but $0 \in (-2, 1)$. Thus, the largest interval is $\boxed{-2 < x < 1}$

2. Water leaks out of a small hole in a cylindrical tank at a rate (cubic feet per hour) which is a constant C times the depth (in feet) of water in the tank. The tank has radius r feet.



(a) The depth $h(t)$ of water in the tank at time t satisfies a differential equation $h' = -kh$ for some constant $k > 0$. Find k in terms of the given data.

We are given that $\frac{dV}{dt} = -Ch$ where $V(t)$ is the volume at time t. But $V(t) = \pi r^2 h$. Hence

$$\pi r^2 \frac{dh}{dt} = -Ch$$

$$\Leftrightarrow \frac{dh}{dt} = -\frac{C}{\pi r^2} h \quad \text{. Thus,} \quad \boxed{k = \frac{C}{\pi r^2}}$$

(b) At noon the depth of water is 10 foot, and at 2:00 P.M. it is 8 foot. Find the depth at time t (i.e., t hours after noon). You can assume here the result of (a): $h' = -kh$.

We know

$$h(0) = 10 \quad \& \quad \frac{dh}{dt} = -kh \quad (*)$$
$$h(2) = 8$$

From $(*)$, we have that

$$h(t) = ce^{-kt} \quad \text{for some constant } c.$$

To satisfy the conditions,

$$10 = c\,e^{-k\cdot 0} \Rightarrow c = 10$$
$$8 = c\,e^{-2k} \Rightarrow \frac{8}{10} = e^{-2k} \Rightarrow k = -\frac{1}{2}\ln(0.8)$$

Thus, $\boxed{h(t) = 10e^{\ln(0.8)\cdot t} = 10(0.8)^t}$