Math 251 Practice Final Exam Fall 2004 Woodward/Kong

PROBLEM 1 Find two unit vectors orthogonal to both $(1, 1, 0)$ and $(1, -1, 1)$.
PROBLEM 2 (a) Let T be the triangle whose vertices are the three points $(0, 0, 1), (2, 0, 0), (0, 3, 0)$. Find the area of T. (b) Find the integral of the function $f(x, y, z) = x$ over T.
PROBLEM 3 (a) Draw a picture of the space curve with parametric equation $r(t) = (2t, \cos(t), \sin(t))$. (b) Using the formula for arclength, find the length of the curve for $t \in [0, T]$. (c) Find the reparametrization of C by arclength, in the direction of increasing t.
PROBLEM 4 (a) Let $f(x, y, z) = x^y + z$ and $P = (1, 1, 1)$. Find the unit vector u pointing in the direction greatest increase of f. (b) Find the rate of increase in the direction of u at P. (c) Find one direction in which the derivative of f at P is zero. (d) Find the equation of the tangent plane to $f(x, y, z) = 2$ at P.
PROBLEM 5 (a) Find the first and second partial derivatives of the function $f(x, y) = xy(2 + x + y)$. (b) Find the critical points of $f(x, y)$. Identify the local maxima, local minima, and saddle points using the second derivative test. (c) Find the tangent plane to $z = f(x, y)$ at $(x, y) = (1, 1)$.
PROBLEM 6 Find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 4$. Draw a picture of the maximum and minimum level sets.
PROBLEM 7 (a)Find the Jacobian of the transformation $x = u + v, y = u - v$. (b) Using the coordinate transformation from part (a), find the integral $\int\int_R e^{x+y} dx dy$ over the region R defined by $|x| + |y| \leq 1$. (Hint: your answer should be positive, since e^{x+y} is a positive function.)
PROBLEM 8 Evaluate the line integral using Green's theorem: $\int_C (x + 2y)dx + (x - 2y)dy$ where C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment from $(1, 1)$ to $(0, 0)$.
PROBLEM 9 (a) Determine whether $F(x, y) = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j}$ is a conservative vector field. If F is conservative, find a function f such that $F = \nabla f$. (c) Evaluate the integral $\int_C F \cdot d\mathbf{r}$ where C is the line segment between $(-1, 1)$ and $(3, 2)$.
PROBLEM 10 (a) State Stokes theorem. (b) Compute the curl of the vector field $F(x, y, z) = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$. (c) Using Stokes theorem compute the integral of $curl(F)$ over the surface S that is the part of the paraboloid $z = 4 - x^2 - y^2$ lying above the plane $z = 0$.
PROBLEM 11 Use the Divergence Theorem to calculate the surface integral $\int\int_S F \cdot d\mathbf{S}$ where $F(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 1$.
PROBLEM 12 (a) Show that if $F = \nabla f$ for some $f = f(x, y, z)$ then $curl(F) = 0$. (b) Use Stokes theorem to show that if $curl(F) = 0$, then F is conservative, that is, the integral $\int_C F \cdot d\mathbf{r}$ depends only on the endpoints of