Practice Final Answers Fall 2008. Note that I have tried to avoid writing column vectors which take too much space. This answer key is meant to check your answers only. The actual exam questions are different, so you should be mainly studying the notes, book etc, not the practice exam. Finally, please note that I also make mistakes, especially when I am in a rush as now, so if something seems not right, possibly there is a mistake, typo etc.

PROBLEM 1 (a) Reduced ref gives $a+b+e=0, c+d=0$. a, c are bound, b, d, e are free. Basis $[-1\ 1\ 0\ 0\ 0], [0\ 0\ -1\ 1\ 0], [-1\ 0\ 0\ 0\ 1]$. (b) Gram-Schmidt gives $[-1\ 1\ 0\ 0\ 0]/\sqrt{2}$, $[0\ 0\ -1\ 1\ 0]/\sqrt{2}$, $[-1\ 0\ 0\ 0\ 1] \mapsto [-1\ 0\ 0\ 0\ 1]-([-1\ 0\ 0\ 0\ 1]\cdot[-1\ 1\ 0\ 0\ 0]/\sqrt{2})[-1\ 1\ 0\ 0\ 0]/\sqrt{2} = [-1/2\ -1/2\ 0\ 0\ 1] \mapsto [-1\ -1\ 0\ 0\ 2]/\sqrt{6}$. (c) $Pv = (v.v_1)v_1+(v.v_2)v_2+(v.v_3)v_3 = ([0\ 0\ 0\ 0\ 1].[-1\ -1\ 0\ 0\ 2]/\sqrt{6})[-1\ -1\ 0\ 0\ 2]/\sqrt{6} = 1/3[-1\ -1\ 0\ 0\ 2]$. (d) The distance of v to V is the distance from v to the closest point in V, that is, Pv, which is $\|v-Pv\| = \|[0\ 0\ 0\ 0\ 1]-1/3[-1\ -1\ 0\ 0\ 2]\| = \|[1/3\ 1/3\ 0\ 0\ 1/3]\| = 1/\sqrt{3}$.

PROBLEM 2 (a) $(-1)^4 3(-70)+(-1)^5 6(-28) = -210-168 = -42$ (b) Subtract the third row from the first and second to create zeros in the first column. Then subtract twice the first from the second. After switching rows, one gets an upper triangular matrix with entries $7, 2, 1, 3$, taking the product gives 42. Hence det is -42.

PROBLEM 3 (a) Each data point gives an equation: $c_0+c_1|-1|+c_2(-1) = 0, c_0+c_1|0|+c_2 0 = 0, c_0+c_1+c_2 = 2$. So the matrix A has entries $[1, 1, -1; 1, 0, 0; 1, 1, 1]$ and b is the vector $[0, 0, 2]$. The solution is $[0\ -1\ 1]$, that is, $f(t) = -|t|+t$. If we don't allow c_2, then the matrix A is $1, 1, 1, 0, 1, 1$. Solving $A^TAx = A^Tb$ gives a solution $f(t) = |t|$. This hits the middle point exactly, is above the left point and below the right point.

PROBLEM 4 (a) false, unless $rref(A) = I$. (b) true, since free variables. (c) True, adding or subtracting rows to other rows does not change det. (d) False, the third is the sum of the first two. (e) true, since $P^3 = (P^2)P = PP = P^2 = P$. (f) True, $A^T = (SDS^{-1})^T = (S^{-1})^T D^T S^T = (S^T)^{-1} D S^T$. (g) true, since $Q^TQ = I$ implies $QQ^T = I$ which means the rows of Q dotted with the rows of Q are either 0 or 1 (if the same row.) (h) True, $\det(A)$ is the product of the eigenvalues, so if zero, one must be zero. (i) Use polar form $r = \sqrt{a^2+b^2}, \theta = \arctan(b/a)$ to get $z = e^{2\pi i/12}$ so that $z^{30} = e^{60\pi i/12} = e^{5\pi i} = -1$. So false. (j) Edge vectors v_1, v_2 are $[-3\ -1], [-3\ -2]$ so area is $1/2|det[v_1, v_2]| = 3/2$.

PROBLEM 5 (a) We have $\lambda_1 = 1, \lambda_1+\lambda_2+\lambda_3 = 0, \lambda_1\lambda_2\lambda_3 = 0$. So the eigenvalues are $1, 0, 0$,. (One could also solve $\det(A-\lambda I) = 0$ here, which takes a little longer.) (b) The eigenvectors are for $\lambda = 1$: the null-space of the matrix with rows $-2/3, 1/3, 1/3; 1/3, -2/3, 1/3; 1/3, 1/3, -2/3$ which reduces to $1, 0, -1; 0, 1, -1; 0, 0, 0$ which is span $1, 1, 1$; and for $\lambda = 0$, the nullspace of the original matrix which is span $1, 0, -1$, and $0, 1, -1$. The problem doesn't say anything about finding an orthogonal diagonalization, so we don't need to make these vectors orthonormal. So S is the matrix of eigenvectors, which is $1, 1, 0; 1, 0, 1; 1, -1, -1$, and D is the matrix of eigenvalues, $1, 0, 0; 0, 0, 0; 0, 0, 0$. (c) Since $A = SDS^{-1}$, A^5 is easy to find: $A^5 = SD^5S^{-1} = SDS^{-1} = A$.

PROBLEM 5' (a) $\det(A-\lambda I) = \lambda^5-1 = 0$ implies that $\lambda^5 = 1$. We solve for λ by using polar form: if $\lambda = re^{i\theta}$then $\lambda^5 = r^5 = e^{5i\theta}$ means that $r^5 = 1$ and $5\theta = 0 \bmod 2\pi$, so $\theta = 0, 2\pi/5, 4\pi/5, \ldots\ldots 8\pi/5$. (b) The nullspace of $A-\lambda I$ is (for any λ) the vector $[1\ \lambda\ \lambda^2\ \lambda^3\ \lambda^4]$. So D is the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi i/5} & 0 & 0 & 0 \\ 0 & 0 & e^{4\pi i/5} & 0 & 0 \\ 0 & 0 & 0 & e^{6\pi i/} & 0 \\ 0 & 0 & 0 & 0 & e^{8\pi i/5} \end{bmatrix}$, and S is the matrix

$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{2\pi i/5} & e^{4\pi i/5} & e^{6\pi i/5} & e^{8\pi i/5} \\ 1 & e^{4\pi i/5} & e^{8\pi i/5} & e^{12\pi i/5} & e^{16\pi i/5} \\ 1 & e^{6\pi i/5} & e^{12\pi i/5} & e^{18\pi i/5} & e^{24\pi i/5} \\ 1 & e^{8\pi i/5} & e^{16\pi/5} & e^{24\pi i/5} & e^{32\pi i/5} \end{bmatrix}$. (c) $A^5 = SD^5S^{-1} = SIS^{-1} = I$. (Incidentally, the matrix S is known as a *Fourier transform*. To say that it comes up a lot in science and engineering is a vast understatement.

PROBLEM 6 A is the matrix with entries $.3, .8; .7, .2$. The trace is $.5 = \lambda_1+\lambda_2$, and $\lambda_1 = 1$ since the columns sum to 1, so that λ_2 must be $-.5$. The eigenvectors are $[8\ 7]$ and $[1-1]$, so D is the matrix with entries $1, 0; 0, -.5$ and S has entries $8, 1; 7, -1$. Computing A^t times the vector $x(0)$ with entries $100, 0$ gives $A^tx(0) = SD^tS^{-1} = [800/15+(-.5)^t 700/15, 700/15-(-.5)^t 700/15]$.

PROBLEM 7 $q(x_1, x_2) = [x_1\ x_2]Q[x_1\ x_2]^T$ where Q is the matrix with entries $2, -2; -2, -1$. We have $\det(Q-\lambda I) = (2-\lambda)(-1-\lambda)-4 = \lambda^2-\lambda-6 = (\lambda-3)(\lambda+2)$. Eigenvectors for $\lambda = -2$ are $nullspace(A-\lambda I) = span[1, 2]$ and eigenvectors for $\lambda = 3$ are $nullspace(-\lambda I) = span[2, -1]$. So $Q = SDS^T$ where D has entries $-2, 0; 0, 3$ and S has entries $1, 2; 2, -1$ all over $\sqrt{5}$. Hence $q(x_1, x_2) = [x_1+2x_2, 2x_1-x_2][-2, 0; 0, 3][x_1+2x_2, 2x_1-x_2]^T/5 = -2(x_1+2x_2)^2/5+3(2x_1-x_2)^2/5 = -2y_1^2/5+3y_2^2/5$. The graph of this function is a saddle (parabola down in the y_1 direction and parabola up in the y_2 direction.

Problem 8: An $n \times n$ matrix is diagonlizable if has n independent eigenvectors. The matrix $[0, 1; 0, 0]$ has one, not two, and so is not. (c) An orthogonal matrix has orthonormal columns. So $[0, 1; 1, 0]$ is orthogonal, but not the identity. (d) The set of vectors $[x_1, x_2]$ with $x_1 x_2 \geq 0$.

Problem 9: (i) If A^2 is invertible, then $\det(A^2) \neq 0$. But $\det(A^2) = \det(A)^2$, so $\det(A)$ is also non-zero, so A is also invertible. (There is also a proof not involving the determinant.) (ii) Suppose $v_1, v_1 - v_2$ are dependent. (By way of contradiction.) Then there is a dependence relation $av_1 + b(v_1 - v_2) = 0$ for some a, b, not both zero. But then $(a + b)v_1 - bv_2 = 0$, and if b is zero then $a + b$ is non-zero. So v_1, v_2 are dependent, which is a contradiction. (iii) If $\lambda = 1$ is an eigenvalue, then $Av = v = v$ for some v. Then $A^2 v = A(Av) = Av = v$ as well, so $\lambda = 1$ is an eigenvalue for A^2. (iv) If A is diagonalizable, then $A = SDS^{-1}$ for some diagonal D and invertible S. Then $A^{-1} = S^{-1}D^{-1}S^{-1}$, and D^{-1} is diagonal and S^{-1} invertible. So $A^{-1} = S_2 D_2^{-1} S_2^{-1}$ for some diagonal D_2 and invertible S_2, that is, A^{-1} is diagonalizable.