NOTES ON LINEAR ALGEBRA AND VECTOR GEOMETRY

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1. Systems of Linear Equations: An example

Here is a simple example of a system of linear equations. Consider the following recipes for regular and light ice cream (which I simplified from Ben & Jerry's Ice-Cream Cookbook.)

	Eggs	Cream (cups)	Sugar (cups)
Regular	2	2	3/4
Light	1	3/2	3/4

Suppose we have 12 eggs, 13 cups cream, and $5\,\frac{1}{4}$ cups sugar. How many pints of regular r and light l can we make without waste? Because each pint of regular needs 2 eggs, the number of eggs needed to make r pints of regular is $2r$. The number of eggs needed to make l pints of light is l. Doing the same for cream and sugar, we get the equations

$$\begin{array}{ccccc} 2r & + & l & = & 12 \\ 2r & + & (3/2)l & = & 13 \\ (3/4)r & + & (3/4)l & = & 5\,\frac{1}{4}. \end{array} \tag{1}$$

This is called a *system of linear equations*. It's called a system because there is more than one equation. It's called linear because each unknown r, l appears *linearly*, in particular, there are no higher powers $r^2, l^2, \ldots$.

It's not hard to solve this system. Subtracting the first equation from the second gives

$$c - e = (1/2)l = 1 \text{ so } l = 2.$$

Plugging $l = 2$ into the first equation gives

$$e = 2r + 2 = 12, \text{ so } r = 5.$$

It's easy to check that the last equation is also solved by $r = 5$. So the solution is

$$r = 5, \quad l = 2.$$

What we have done is a simple case of *elimination*. We'll come back to this later.

2. Vectors

Any linear system of equations can be thought of both algebraically and geometrically, using *vectors*. After introducing vectors, we will come back to talk about the geometry of the ice-cream system (1).

2.1. **Two-vectors.** A 2-vector is a pair of numbers.

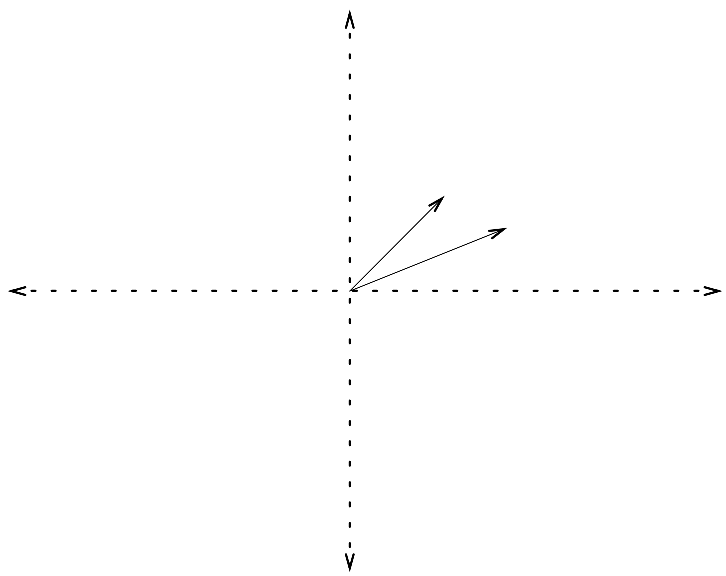
Example 2.1. $\mathbf{v} = [5 \;\; 2]$ and $\mathbf{w} = [3 \;\; -3]$ are 2-vectors.

Two vectors are equal if they have the same components, in order. If we write **v**horizontally, as above, **v** is called a *row vector*. If we write **v** vertically

$$\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

v is a *column vector*. It's still the same vector, however we write it.

Geometrically, we represent a 2-vector $\mathbf{v} = [v_1 \;\; v_2]$ as an arrow in the plane. The *tail* of the vector can be at any point. The head of the vector v_1 units to the right, and v_2 units above the tail of the vector. Usually, we draw the vector with tail at $(0,0)$. Here is are the vectors [5 2] and [3 3].



We add 2-vectors by adding their components:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5+3 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}.$$

Subtraction is similar:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5-3 \\ 2-3 \end{bmatrix} \begin{bmatrix} 2 \\ -1. \end{bmatrix}$$

Subtraction might come up in the following situation. You go to the store and buy 5 pints of regular and 2 pints of light, and add it to what ice-cream is already in the fridge, which is your room-mates. That evening, while you're not looking, your little brother eats 3 pints of regular and 3 pints of light. There are now 2 more pints of regular, and 1 less pint of light, than were there to begin with! He better eat -1 more pint of light ice-cream - otherwise you're going to be in trouble!

We can multiply vectors by numbers (*scalars*); this operation is called *scalar multiplication.*

Example 2.2.

$$2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 \\ 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}.$$

The magnitude of $\mathbf{v} = [v_1 \quad v_2]$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}.$$

This is the distance from the head to the tail. If the vector has magnitude 1, it is called a *unit vector*. For any non-zero vector $\mathbf{v}$ there is a *unit vector in the direction* of v,

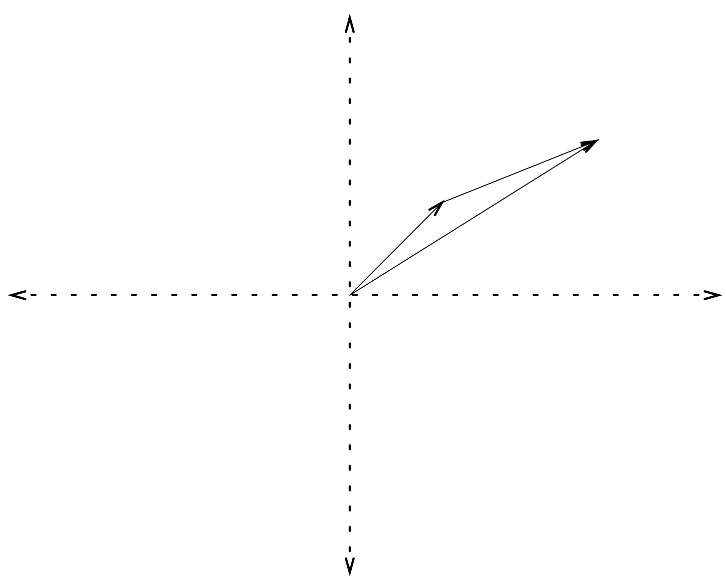
$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

Example 2.3. The unit vector in the direction of $\mathbf{v} = [5 \quad 2]$ is

$$\begin{aligned}\mathbf{u} &= \frac{1}{\sqrt{5^2+2^2}}[5 \quad 2] \\ &= \frac{1}{\sqrt{29}}[5 \quad 2] \\ &= \left[\frac{5}{\sqrt{29}} \quad \frac{2}{\sqrt{29}}\right].\end{aligned}$$

Vectors can be added geometrically by putting them *head to tail*. Say $\mathbf{v} = [v_1 \quad v_2]\,, \mathbf{w} = [w_1 \quad w_2]$ are 2-vectors. Draw $\mathbf{v}$ with tail at 0, and $\mathbf{w}$ with tail at the head of $\mathbf{v}$. Now draw the vector from tail $(0,0)$ to the head of $\mathbf{w}$. This is the vector $\mathbf{v} + \mathbf{w}$.

Here is the vector [5 2] drawn head to tail with [3 3].



2.2. **Three-vectors.** Now let's look at 3-vectors. A 3-vector $\mathbf{v}$ is a triple of numbers, called the components of $\mathbf{v}$.

Example 2.4.

$$\mathbf{v_r} = \begin{bmatrix} 2 \\ 2 \\ 3/4 \end{bmatrix}, \quad \mathbf{v_l} = \begin{bmatrix} 1 \\ 3/2 \\ 3/4 \end{bmatrix}$$

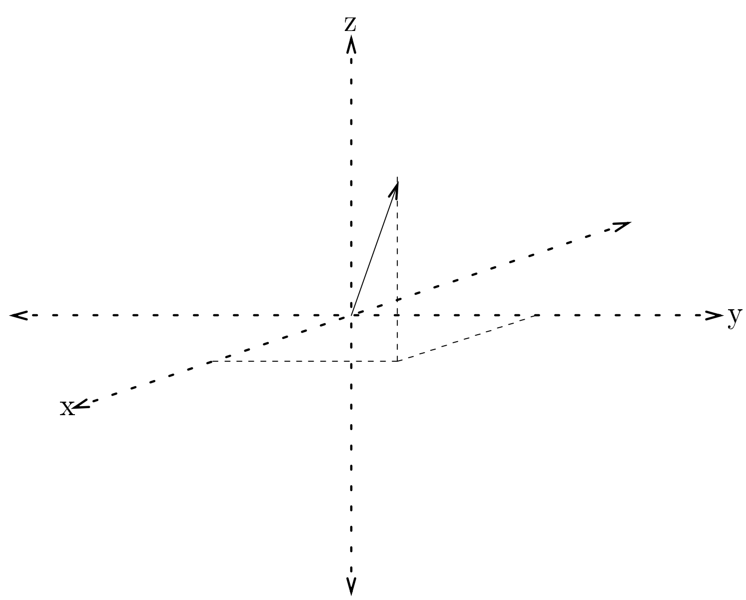
are column 3-vectors. We can multiply 3-vectors by scalars:

$$5\mathbf{v_r} = \begin{bmatrix} 10 \\ 10 \\ 15/4 \end{bmatrix}, \quad 2\mathbf{v_l} = \begin{bmatrix} 2 \\ 3 \\ 6/4 \end{bmatrix}.$$

We can add column vectors by adding their components:

$$5\mathbf{v_r} + 2\mathbf{v_l} = \begin{bmatrix} 10+2 \\ 10+3 \\ 15/4+6/4 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}.$$

Recall that a point in space is a triple of numbers $x = (x_1, x_2, x_3)$. A 3-vector $\mathbf{v} = [v_1\ v_2\ v_3]$ is geometrically represented by an arrow. If the tail is the point $x = (x_1, x_2, x_3)$, the head of the vector is at $(x_1 + v_1, x_2 + v_2, x_3 + v_3)$. Usually, we draw the vector with tail at $(0, 0, 0)$. Here is the vector $[1\ 2\ 3]$ with tail at $(0, 0, 0)$.



Multiplication by a scalar leaves the direction unchanged, but changes the length.

$$2\,[1\ 2\ 3] = [2\ 4\ 6]\,.$$

Addition is head to tail, just as for 2-vectors. The length of $\mathbf{v} = [v_1\ v_2\ v_3]$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{2}$$

2.3. ***n*-vectors.** An n-vector is an n-tuple of numbers

$$\mathbf{v} = [v_1\ v_2 \ldots v_n]\,.$$

We say that $\mathbf{v}$ has *size* n. Everything we've done so far works for n-vectors also. For example, the length of the vector $\mathbf{v} = [v_1\ v_2\ \ldots\ v_n]$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

2.4. **Dot products.** Now let's define multiplication of vectors, called *dot product*. Dot product is unlike other products you may have met because the product of two vectors is not another vector, but instead a number. The dot product of a vector with another vector is only defined if they are the same size. Suppose

$$\mathbf{u} = [u_1\ u_2\ \ldots u_n]\,, \quad \mathbf{v} = [v_1\ v_2\ \ldots v_n]\,.$$

Then we define

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

For instance,

$$[1\ 2\ 3] \cdot [4\ 5\ 6] = 1(4) + 2(5) + 3(6) = 32.$$

Here are the reasons it makes sense to call this multiplication. First of all, a 1-vector is just a number. In this case there is no sum,

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1$$

which is just ordinary multiplication. Second, if we take the product of $\mathbf{u}$ with itself we get

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \ldots + u_n^2 = \|u\|^2$$

which is the square of the length of $\mathbf{u}$, defined in (2). This is just like ordinary multiplication, since for any number x we have

$$x \cdot x = |x|^2.$$

Geometrically, the dot product has to do with the angle θ between the two vectors:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta). \tag{3}$$

This formula can be proved using the law of cosines. Look at the triangle with edge vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$. The law of cosines says

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|u\|\|v\| \cos(\theta).$$

The left-hand-side is

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}.$$

Subtracting the quantity

$$\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

from both sides leads to the formula.

The equation (3) can be used to compute the angle between vectors:

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

Example 2.5. The angle between the vectors $\mathbf{u} = [1\ 1\ 0]\,, \quad \mathbf{v} = [0\ 1\ 1]$ is

$$\begin{aligned} \theta &= \arccos\left(\frac{1(0) + 1(1) + 1(1)}{\sqrt{1^2 + 1^2 + 0^2}\sqrt{0^2 + 1^2 + 1^2}}\right) \\ &= \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}. \end{aligned}$$

The formula (3) has an important special case. Two vectors are *prependicular* or *orthogonal* if the angle between them is $\pi/2$, that is, 90 degrees. This is the case if and only if

$$\theta = \frac{\pi}{2} \iff \cos(\theta) = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

Two vectors are perpendicular if and only if their dot product is zero.

Example 2.6. Suppose we want to find a vector $\mathbf{v} = [v_1\ v_2\ v_3]$ perpendicular to $\mathbf{u} = [1\ 1\ 1]$. The condition that the dot product is zero is

$$1(v_1) + 1(v_2) + 1(v_3) = v_1 + v_2 + v_3 = 0.$$

It's easy to find solutions. For example, $v_1 = 1, v_2 = -1, v_3 = 0$, or $v_1 = 0, v_2 = 1, v_3 = -1$ are both solutions. These give the vectors

$$\mathbf{v} = [1\ -1\ 0]\,,\ \text{or}\ \mathbf{v} = [0\ 1\ -1]\,.$$

2.5. **Properties of Vector Operations.** Here are some of the properties of vector operations defined in this section. The main reason to emphasize them is that later we will define new operations which do *not* satisfy some of these axioms. So we have to be very careful!

(1)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(Vector Addition is Commutative)
(2)	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	(Vector Addition is Associative)
(3)	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	(Scalar Mult. is Distributive)
(4)	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	(Dot Product is Commutative)
(5)	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	(Dot Product is Distributive
(6)	$\mathbf{v} \cdot \mathbf{v} = \lVert\mathbf{v}\rVert^2$	(Dot Self-Product is Square Length)

2.6. **Geometry of the ice-cream system.** Finally let's come back the geometry of the ice-cream system. We can replace the three equalities in the system by one vector equality (since vectors are equal if and only if their components are equal.)

$$\begin{bmatrix} 2r + l \\ 2r + (3/2)l \\ (3/4)r + (3/4)l \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}.$$

Using scalar multiplication and vector addition we get the vector form of the system:

$$r \begin{bmatrix} 2 \\ 2 \\ 3/4 \end{bmatrix} + l \begin{bmatrix} 1 \\ 3/2 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}.$$

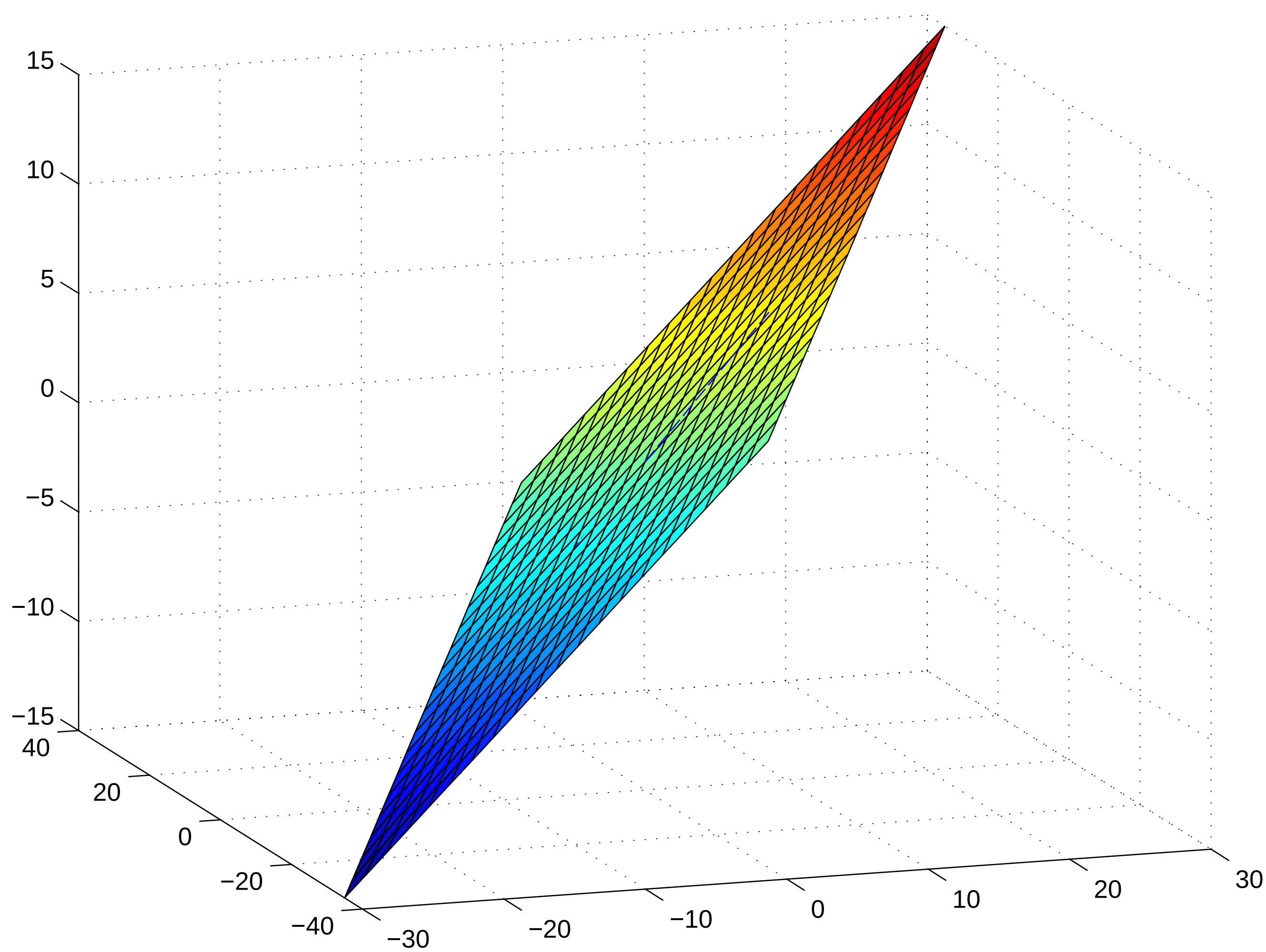
This is the same set of equations, re-written using vectors. What do we see by looking at the system geometrically, that we didn't see before by looking at it algebraically? For one thing, it was very lucky that we were able to solve the system. Suppose that we had slightly different ingredients. Would we still have been able to use up the ingredients without waste? Let's define

$$\mathbf{v_i} = \begin{bmatrix} 12 \\ 12 \\ 21/4 \end{bmatrix}$$

the vector of ingredients. To say that we can use up these ingredients without waste means that we can find numbers r and l such that

$$r\mathbf{v}_r + l\mathbf{v}_l = \mathbf{v}_i.$$

The set of all vectors of this form lie in a plane P - the set of ingredient vectors that can be used up without waste. The head of $\mathbf{v}_i$ is the point reached by moving r times along $\mathbf{v}_r$, and l times $\mathbf{v}_l$. As r and l vary, we get any vector whose direction is a "combination" of the directions of $\mathbf{v}_r$ and $\mathbf{v}_l$. This is why P is called the plane *spanned* by $\mathbf{v}_r$ and $\mathbf{v}_l$. See the figure below.



Of course, if you are actually doing the cooking, it might be nice to have something left after your finished - so you might try to avoid landing on the plane P!

3. Matrices

In the last section, we wrote the ice-cream system in the simple form

$$\mathbf{v_i} = r\mathbf{v_r} + l\mathbf{v_l}.$$

Now we'll see how to write it even more simply, using *matrices*. A matrix is a table of numbers, called the *entries* of the matrix. If a matrix has m rows and n columns, it is called an $m \times n$ matrix. The entry in the i-th row and j-th column is called the ij-th entry. For instance, the matrix for the ice-cream system is the 2×3 matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix}.$$

One can think of a matrix as a collection of rowvectors, or as a collection of column vectors. The row vectors for the matrix A are

$$[2\ 1]\,,\ [2\ 3/2]\,,\ [3/4\ 3/4]\,.$$

The column vectors are

$$\begin{bmatrix} 2 \\ 2 \\ 3/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3/2 \\ 3/4 \end{bmatrix}.$$

Here are a few of the many special kinds of special matrices.

(1) The $m \times n$ zero matrix $\mathbf{0}_{mn}$, whose entries are all zero. For example,

$$\mathbf{0}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If there is no confusion about the size, we drop the subscripts and write $\mathbf{0}$ for the zero matrix.

(2) A matrix is *square* if it has the same number of rows as columns. For instance,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a square matrix, more precisely 2×2.

(3) A square matrix is *diagonal* if the only non-zero entries are on the diagonal, for instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is diagonal.

(4) A square matrix A is *upper triangular* if all of the entries below the diagonal are zero. For instance,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

is upper triangular. A is *lower triangular* if all of the entries above the diagonal are zero. A is strictly upper (or lower) triangular if it is upper (or lower)triangular and all of the diagonal entries are zero. For instance,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

is strictly lower triangular.

(5) The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose columns are the rows of A, and whose rows are the columns of A. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

If a matrix A is equal to its own transpose A^T, it is called *symmetric*. For instance,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is a 3×3 symmetric matrix.

(6) A matrix is a *permutation matrix* if there is exactly one 1 in each row and each column, and otherwise the matrix is zero. For example,

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is a permutation matrix.

Example 3.1. Show that if P is a permutation matrix, then so is P^T.

3.1. **Matrix addition, subtraction, and scalar Multiplication.** Matrices are added or subtracted in the same way as vectors, by adding or subtracting entries. For example,

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 3 & 7 \\ 6 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

We can multiply scalars times matrices by multiplying each entry by the scalar.

$$2\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 6 \\ 8 & 10 \end{bmatrix}.$$

3.2. **The product of a matrix and a column vector.** Let's start with an example. Suppose A is a 3×2 matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2$. Suppose $\mathbf{x}$ is the column vector

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Then the product $A\mathbf{x}$ is the sum

$$5\mathbf{v}_1 + 2\mathbf{v}_2.$$

That is, the product of a matrix times a vector is a sum of the column vectors of the matrix, with coefficients given by the components of the vector. For instance,

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} &= 5\begin{bmatrix} 2 \\ 2 \\ 3/4 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3/2 \\ 3/4 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}. \end{aligned}$$

Here is an example with a 2×2-matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Multplying this matrix times a vector has the effect of switching the first and second components!

We can re-write the ice-cream equations a second time using this product. The equations (1) are written in matrix form

$$\begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix} \begin{bmatrix} r \\ l \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}.$$

There is another way of looking at the product of a matrix A times a vector $\mathbf{x}$, using dot products. The components of the product are the dot products of the rows of A with the vector $\mathbf{x}$. For instance,

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} &= 5 \begin{bmatrix} 2 \\ 2 \\ 3/4 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3/2 \\ 3/4 \end{bmatrix} \\ &= \begin{bmatrix} 5(2) + 2(1) \\ 5(2) + 2(3/2) \\ 5(3/4) + 2(3/4) \end{bmatrix}. \end{aligned}$$

The first component is the dot product of $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ with [2 1], the second component is the dot product with [2 3/2] and so on.

In general notation, suppose A has column vectors $\mathbf{v}_1, \ldots, \mathbf{v_n}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix},$$

Define

$$A\mathbf{x} = x_1 \mathbf{v}_1 + \ldots + x_n \mathbf{v_n}.$$

If A has row-vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{w}_1 \cdot \mathbf{x} \\ \mathbf{w}_2 \cdot \mathbf{x} \\ \\ \\ \mathbf{w}_m \cdot \mathbf{x} \end{bmatrix}$$

is the vector of dot products.

For each square size, there is a special matrix, called the *identity matrix* I which has 1's along the diagonal and $0'$ everywhere else. For instance, the 3×3 identity is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity matrix has the property that I times any vector $\mathbf{v}$ is itself:

$$I\mathbf{v} = \mathbf{v}$$

For instance, using the definition we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}.$$

3.3. **Matrix products.** To define the product of two matrices A and B, we take the product of the matrix A times each column of B. These new vectors form the columns of the matrix, denoted AB. Because each column is made up of dot products of rows of A with columns of B, the matrix product is the matrix of dot products. For example,

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} [1\ 2] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [12] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ [2\ 3] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [2\ 3] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ [3\ 4] \begin{bmatrix} -1 \\ 1 \end{bmatrix} & [3\ 4] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Notice that the matrix product only makes sense if the dot products of the rows of A with the columns of B make sense. In other words, the rows of A have to be the same size as the columns of B. To put it one more way, if A is an $m \times n$ matrix and B is a $p \times q$ matrix, the product AB makes sense only if $n = p$. The result is an $m \times q$ matrix.

In general notation let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be the row-vectors of A, and $\mathbf{v}_1, \dots, \mathbf{v}_q$ the column vectors of B. The matrix product AB is the matrix whose ij-th entry is the dot product $\mathbf{v}_i \cdot \mathbf{v}_j$.

3.4. **Matrix multiplication is counter-intuitive.** Matrix multiplication is counter-intuitive in a number of different ways.

(1) Matrix multiplication is *not* commutative, that is AB is not necessarily the same matrix as BA, even if both are defined. For AB to be defined A must have the same number of columns as B has rows. For BA to be defined, B has the same number of columns as A has rows. Here is an example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix},$$

which is not equal to

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$

(2) Just because $AB = 0$ doesn't mean that $A = 0$ or $B = 0$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

but neither of the matrices (which are equal) are zero. In fact, this property isn't true for vectors either. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors of the same size, and $\mathbf{v} \cdot \mathbf{w} = 0$. As we said before, this means that $\mathbf{v}$ is perpendicular to $\mathbf{w}$, not that either $\mathbf{v}$ or $\mathbf{w}$ is zero.

(3) If $AB = AC$ and A is non-zero, then it is not necessarily true that $B = C$. We can't just divide A from both sides, since the expression $1/A$ doesn't make sense - yet.

(4) If A is a square matrix we can define it's matrix powers
$$A^2 = A\,A,\ A^3 = A\,A\,A,$$
et cetera. If $A^2 = 0$, this doesn't mean that A is zero. For instance,
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
has $A^2 = 0$, but A is not zero. This property *is* true for vectors and dot product:
$$\mathbf{v} \cdot \mathbf{v} = 0 \implies \|\mathbf{v}\|^2 = 0 \implies \mathbf{v} = 0$$
since the only vector with length zero is the zero vector.

3.5. **Properties of matrix operations.** Here are some properties that matrix addition, multiplication, and transpose *do* have.

(1)	$A + B = B + A$	(Commutativity of Addition)
(2)	$A + (B + C) = (A + B) + C$	(Assoc. of Addition)
(3)	$A(BC) = (AB)C$	(Assoc. of Matrix Product)
(4)	$A(B + C) = AB + AC$	(Distrib. of Left Matrix Product)
(5)	$(A + B)C = AC + BC$	(Distrib. of Right Matrix Product)
(6)	$(A^T)T = A$	(Transpose is an Involution)
(7)	$(AB)^T = B^T A^T$	(Tranpose of a Product Changes Order)

Properties (1), (2), (4), (5), and (6) are all easy. Property (3) is best explained later, when we talk about linear transformations. Property (7) is justified as follows:
$$\begin{aligned} ij\text{-th entry of } (AB)^T &= ji\text{-th entry of } AB \\ &= (\text{row } j \text{ of } A)\cdot\ (\text{column } i \text{ of B}) \\ &= (\text{column } j \text{ of } A^T) \cdot (\text{row } i \text{ of } B^T) \\ &= (\text{row } i \text{ of } B^T)\cdot\ (\text{column } j \text{ of } A^T) \\ &= ij\text{-th entry of } B^T A^T. \end{aligned}$$

3.6. **Matrix products in applications.** Let's go back to the ice cream example to illustrate the meaning of matrix multiplication. The matrix A that multiplies the number of pints vector to give the ingredients-needed vector is the matrix
$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix}.$$
That is,
$$A \begin{bmatrix} r \\ l \end{bmatrix} = \begin{bmatrix} e \\ c \\ s \end{bmatrix}.$$

Suppose we look at the *cost* and *weight* of the ingredients (taking round numbers for the sake of discussion):

ingredient	cost (cents)	weight (g)
eggs	10	50
cream	80	50
sugar	10	40

Let B be the matrix

$$B = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix}.$$

The matrix B transforms the ingredient vector into the cost/weight vector:

$$B \begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$$

where t is the number of cents that the ingredients cost and w is their weight. For instance, the cost/weight of the ingredients we had before is

$$B \begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix} \begin{bmatrix} 12 \\ 13 \\ 5\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 10(12) + 80(13) + 10(5\frac{1}{4}) \\ 50(12) + 50(13) + 50(5\frac{1}{4}) \end{bmatrix} = \begin{bmatrix} 1212.5 \\ 1512.5 \end{bmatrix}.$$

That is, the 12 eggs, 13 cups cream and $5\frac{1}{4}$ sugar costs \$12.13 (the cream costs the most) and weighs 1.51 kilograms.

Now suppose we want a matrix that takes the regular/light vector to the cost/weight vector. This is the role of the matrix product:

$$BA \begin{bmatrix} r \\ l \end{bmatrix} = B \begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}.$$

The matrix product is

$$BA = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3/2 \\ 3/4 & 3/4 \end{bmatrix} = \begin{bmatrix} 217.5 & 157.5 \\ 230 & 155 \end{bmatrix}.$$

vThe 11-entry is the cost of a pint of light; the 12-entry is the cost of a point of regular. The second row records the weight of a pint of regular or light.

4. Elimination

4.1. **Row operations.** Elimination is the procedure by which we try to solve a system of linear equations by subtracting multiples of the equations from each other to eliminate the unknowns.

Example 4.1. To solve the ice-cream system we performed the following steps. Subtract the first equation from the second in

$$\begin{array}{rcrcl} 2r & + & l & = & 12 \\ 2r & + & (3/2)l & = & 13 \\ (3/4)r & + & (3/4)l & = & 5\,\frac{1}{4}. \end{array} .$$

to get

$$\begin{array}{rcrcl} 2r & + & l & = & 12 \\ & & \frac{1}{2}l & = & 1 \\ (3/4)r & + & (3/4)l & = & 5\,\frac{1}{4}. \end{array} .$$

Multiply the second equation by two to get

$$\begin{array}{rcrcl} 2r & + & l & = & 12 \\ & & l & = & 2 \\ (3/4)r & + & (3/4)l & = & 5\,\frac{1}{4}. \end{array}$$

Now substitute $l = 2$ into the first and third, and solve for r.

We can do the same steps in matrix form, using a little less ink. The *augmented matrix* for the ice-cream system (1) is

$$\left[\begin{array}{cc|c} 2 & 1 & 12 \\ 2 & 3/2 & 13 \\ 3/4 & 3/4 & 5\,\frac{1}{4}. \end{array}\right].$$

to get

$$\left[\begin{array}{cc|c} 2 & 1 & 12 \\ 0 & 1/2 & 1 \\ 3/4 & 3/4 & 5\,\frac{1}{4}. \end{array}\right].$$

Multiply the second equation by two to get

$$\left[\begin{array}{cc|c} 2 & 1 & 12 \\ 0 & 1 & 2 \\ 3/4 & 3/4 & 5\,\frac{1}{4}. \end{array}\right].$$

The second equation says $0r + 1l = 2$, that is, $l = 2$. Substituting into the first equation gives $r = 5$, as before.

Example 4.2. Suppose we want to solve the system of three equations with three unknowns

$$\left[\begin{array}{rcrcrcl} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ -x & & & + & z & = & 3 \end{array}\right].$$

The matrix form of the system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ -1 & 0 & 1 & 3 \end{array}\right]$$

In the first step we add equation to equation 1 to equation 3 to get (also written in matrix form on the right)

$$\left[\begin{array}{rcrcrcl} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ & - & y & + & z & = & 4 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 4 \end{array}\right].$$

Then we add equations 2 and 3 to get

$$\left[\begin{array}{rcrcrcl} x & - & y & & & = & 1 \\ & & y & - & z & = & 2 \\ & & & & 0 & = & 6 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right].$$

The last equation $0 = 6$ is a contradiction (obviously wrong). This means that the system has no solutions.

There are three possible *row operations*, or moves in elimination:

(1) Add a multiple of one row (equation) to another.
(2) Multiply a row (equation) by a non-zero number.
(3) Switch two rows (equations).

4.2. **Row-echelon and reduced row-echelon form.** Elimination can stop when the matrix is in *row-echelon form*:

Definition 4.3. (1) All rows of zeroes are at the bottom.
(2) The first non-zero entry in any row is a 1, called a *leading* 1 or *pivot*.
(3) The leading 1 in any row is to the right of the leading 1's above it.

If, in addition, the entries above any leading 1 are zero, the matrix is said to be in *reduced row-echelon form*.

Let's solve the system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 6 & 8 & 1 \end{array}\right].$$

We subtract 2 times row 1 from row 2, and 3 times row 1 from row 3 to get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right].$$

Now we switch rows 2 and 3 and multiply by -1 to get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The augmented matrix is now in row-echelon form. To solve it, we write the equations for the matrix and do back-substitution. The equations are

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ & & & & z & = & 2 \\ & & & & 0 & = & 0 \end{array}.$$

Substituting $z = 2$ into the first equation gives $x + 2y = -5$ or $x = -5 - 2y$. The solution set to this system is therefore

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad z = 2 \text{ and } x = -5 - 2y \right\}.$$

In other words, the solution set is

$$\left\{ \begin{bmatrix} -5 - 2y \\ y \\ 2 \end{bmatrix} \right\}.$$

This system has an infinite number of solutions, one for each value of y. The variables x, z that correspond to columns containing leading 1's are called the *bound variables*; the other variables are called *free variables.* The bound variables can be expressed in terms of the free variables, using the equations corresponding to the rows of the row-echelon form system and back substitution.

Instead of back substitution, we can do further row operations to get the matrix into reduced row echelon form. For instance, subtracting three times the second row from the first in

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

gives

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

4.3. **The number of solutions to a linear system.** The three examples we have done so far show the three possible outcomes of elimination:

(1) There is a unique solution if the system is consistent and there is a leading 1 in every column (to the left of the bar).
(2) There are infinite solutions if the system is consistent and there are some columns without leading 1's.
(3) The system is inconsistent if there is a row of zeroes, and a non-zero number to the right of the bar. In this case there are no solutions.

Let's do another example with an infinite number of solutions.

Example 4.4. We solve the system

$$\begin{array}{ccccccccc} 2x & + & 6y & - & 2z & + & 2w & = & 6 \\ x & + & 3y & + & y & + & 10z & = & 15 \end{array}.$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 2 & 6 & -1 & 2 & 6 \\ 1 & 3 & 1 & 10 & 15 \end{array}\right].$$

Switch the first and second rows to get a leading one in the first column:

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 2 & 6 & -1 & 2 & 6 \end{array}\right].$$

Subtract twice the first row from the second to get

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 0 & 0 & -3 & -18 & -24 \end{array}\right].$$

Multiply the second row by $-1/4$ to create a leading one in the second row.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 10 & 15 \\ 0 & 0 & 1 & 6 & 8 \end{array}\right].$$

The matrix is in row-echelon form. At this point we could write out the equations and do back-substitution to find the answer. Instead, we keep going to reduced row-echelon form. Subtract the second row from the first to get

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 7 \\ 0 & 0 & 1 & 6 & 8 \end{array}\right].$$

Now this is reduced row-echelon form. The equations are

$$\begin{array}{ccccccccc} x & + & 3y & & & + & 4w & = & 7 \\ & & & & z & + & 6w & = & 8 \end{array}.$$

The leading 1's are in the first and third rows. So the bound variables are x and z; the free variables are y and w. Write the bound variables in terms of the free variables

$$x = 7 - 3y - 4w, \quad z = 8 - 6w.$$

The solution set to the system is all vectors satisfying these equations. Since y and w can be anything, the solution set can be written

$$\left\{ \begin{bmatrix} 7 - 3y - 4w \\ y \\ 8 - 6w \\ w \end{bmatrix} \right\}.$$

Here we have substituted the formulas for the bound variables. Since there are free variables, there are an infinite number of solutions.

The fact that there are always $0, 1$ or infinite solutions is a special feature of linear systems of equations. Non-linear equations, can have other numbers of solutions, for example. For example $x^2 = 1$, has two solutions $x = \pm 1$.

Note 4.5.
(1) It's not possible to read off the number of solutions from the number of unknowns and the number of equations, without doing the elimination.
(2) If the number of unknowns is greater than the number of equations, some of the unknowns must be free. So there are always infinite or no solutions.
(3) The system is called homogeneous if the numbers to the right of the equals signs/bar are all zero. Since these systems are always consistent, the number of solutions is infinity or one.

4.4. **Application to polynomial interpolation.** There is a unique line passing through the points $(1, 1)$ and $(-1, 1)$. How many degree two polynomials are there passing through these points? [1] Although this problem seems non-linear (a parabola is a graph of a quadratic function) in fact this is a system of linear equations. Suppose that the function is

$$f(x) = ax^2 + bx + c.$$

The unknowns here are the values of a, b, c, since we are solving for the parabola, not x. Each point gives us an equation for a, b, c:

$$a(1)^2 + b(1) + c = 1$$
$$a(-1)^2 + b(-1) + c = 1.$$

[1] If you want, imagine you have collected some experimental data and know for some reason that the quantities you are measuring are related by a degree two function.

The matrix form of this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right].$$

Since the number of unknowns is greater than the number of equations, from Theorem 4.5 we know that there are either zero or infinite solutions. To figure out which, we have to do elimination. There is already a leading 1 in the first row, so there's nothing to do there. We subtract the first row from the second to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right].$$

Divide the second row by -2 to get a leading 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

This matrix is now in row-echelon form. To get reduced row-echelon form, subtract the second row from the first:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

The equations are

$$a + c = 1, \quad b = 0.$$

The bound variables are a, b; the free variable is c. Expressing the bound variables in terms of the free variables gives

$$a = 1 - c, \quad b = 0.$$

The solution vectors are

$$\begin{bmatrix} 1-c \\ 0 \\ c \end{bmatrix}.$$

The solution functions are

$$f(x) = (1-c)x^2 + 0x + c = (1-c)x^2 + c.$$

It's easy to check that these all satisfy $f(\pm 1) = 1$. Since there is one solution for each value of c, there are infinite solutions.

Example 4.6. It's easy to come up with a similar example which has *no* solutions: If the data points are $(1, 1)$ and $(1, -1)$, there is no function with these values because any function takes only one value at any value of x. If we try to solve this system, we get the matrix form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

which has rref (reduced row-echelon form)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

This is inconsistent.

Example 4.7. Find all polynomials of degree 2 passing through the points $(-1,1),(0,2),(-1,1)$. The resulting system of equations has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

The matrix has rref equal to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

There is a unique solution, $f = -x^2 + 2$.

Here is the general result, which we will prove later using determinants:

Theorem 4.8. *Given n points $(x_1, y_1), \dots, (x_n, y_n)$, with $x_1, \dots, x_n$ distinct, there is a unique polynomial of degree $n+1$ passing through them. If $d > n+1$, there are infinitely many polynomials passing through these points.*

5. Matrix inverses

We said before that $AD = AE$ does not imply $D = E$, even if A is non-zero. This is because it doesn't make any sense to "divide by A on both sides", for arbitrary matrices. The matrices for which it does make sense are called *invertible*.

5.1. The definition of the inverse.

Definition 5.1. A matrix A is left invertible if there is a matrix B such that $BA = I$. A matrix A is right invertible if there is a matrix C such that $AC = I$. A matrix is invertible if it is both left and right invertible.

Note 5.2. (1) If A is both left and right invertible then the left and right inverses are equal:

$$C = IC = (BA)C = B(AC) = BI = B.$$

In this case the inverse is unique, by the same argument. The left/right inverse is called the *inverse* of A and denoted A^{-1}.

(2) If A is left invertible, then $AD = AE$ does imply that $D = E$, since we can multiply both sides by the left inverse B:

$$BAD = BAE \implies ID = IE \implies D = E.$$

(3) Similarly, for right invertible matrices $DA = EA$ implies $D = E$.

Example 5.3. The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is invertible with inverse $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$. More generally, any diagonal matrix with diagonal entries $a_{11}, \dots, a_{nn}$ is invertible with inverse the diagonal matrix with entries $1/a_{11}, \dots, 1/a_{nn}$.

Example 5.4. The matrix $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is invertible with inverse $A^{-1} = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$.

Example 5.5. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ has right inverse $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$.

There is a simple formula for the inverse of a 2×2 matrix. The *determinant* of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant is non-zero if and only if the matrix is invertible; the a formula for the inverse is (check!)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Later we'll generalize this formula to larger matrices.

If A is invertible then so are its square A^2 and its transpose:

$$(A^2)^{-1} = (A^{-1})^2$$

since $A^2(A^{-1})^2 = AAA^{-1}A^{-1} = AIA^{-1} = I$, and

$$(A^T)^{-1} = (A^{-1})^T.$$

Similarly, the inverse of A^n is $(A^{-1})^n$, for any $n > 0$. If A and B are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

You can think of the reason for this in the following silly way. Suppose at the beginning of the day you put on your shoes (call this operation A) and tie your shoe laces (call this operation B). What do you do when you come home? This is a classic example of operations that do not commute.

5.2. **Finding inverses via elimination.** The most efficient way of finding the inverse of a square matrix A is via elimination. Consider the vector equation $A\mathbf{x} = \mathbf{y}$. The inverse matrix A^{-1} solves the equation $\mathbf{x} = A^{-1}\mathbf{y}$. So if we can express $\mathbf{x}$ in terms of $\mathbf{y}$, we can read off the coefficients to get the matrix A^{-1}.

Writing out the equations for $A\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n. \end{aligned}$$

We now have a system of linear equations with variables on the right-hand side. Reading off the coefficients we get the matrix form

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ & & & & & & \vdots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & 0 & \ldots & 1 \end{array}\right].$$

or, for short, $[A|I]$. To solve for $\mathbf{x}$, we do elimination. If, at the end, we get the identity matrix on the left-hand side, then the right-hand side is the inverse is the matrix on the right.

Example 5.6. To find the inverse of $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ we do elimination on

$$\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{array}\right].$$

We divide the first row by 2 and the second by 3 to get the rref

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right].$$

The inverse is $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$.

Example 5.7. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$. We do elimination on

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{array}\right].$$

We subtract twice the first row from the second to get

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right].$$

The second equation is inconsistent: this matrix has no inverse.

Example 5.8. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. We do elimination on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We subtract the first row from the second to get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We subtract the second from the first and third to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right].$$

Multiply the third by -1 to create a leading 1:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

Now add the third to the first, and subtract twice the third from the second to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

The inverse is

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Notice that if A is invertible, then $A\mathbf{x} = \mathbf{y}$ has a unique solution for every $\mathbf{y}$. This implies that $\mathrm{rref}(A) = I$, since (1) if there were a row of zeroes, the system would be inconsistent for some values of y, and (2) if column did not contain a leading 1, there would be free variables, so any solution would not be unique. So *any invertible matrix is automatically square.* Let's summarize what we've shown so far:

Theorem 5.9. *A matrix A is invertible if and only if A is square and $\mathrm{rref}(A) = I$. In this case, the inverse is the right hand side of the matrix $\mathrm{rref}([A|I])$.*

It's easy to check that a 2×2-matrix A has $\mathrm{rref}(A) = I$ if and only if the determinant is non-zero.

5.3. **Application: A formula for the line between two points.** There is a unique line through any two points (x_1, y_1), (x_2, y_2). Let's find a formula for it, using matrices. (Think for a moment about you would find a formula another way.) The equation for a line is $f(x) = ax + b$. The two data points give

$$ax_1 + b = y_1, \quad ax_2 + b = y_2$$

or in matrix form

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let's find the inverse using the formula for two by two inverses. The determinant is $x_1 - x_2$, so the inverse is

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix}^{-1} \frac{1}{x_1 - x_2} \begin{bmatrix} x_2 & -1 \\ x_1 & -1 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{x_1 - x_2} \begin{bmatrix} y_1 - y_2 \\ -y_1 x_2 + x_1 y_2 \end{bmatrix}$$

or

$$f(x) = \frac{y_1 - y_2}{x_1 - x_2} x + \frac{x_1 y_2 - y_1 x_2}{x_1 - x_2}.$$

Check that the slope and y-intercept make sense.

6. Determinants

Determinants are another way, besides finding the rref, of determining whether a matrix is invertible. The determinant is non-zero if and only if the matrix is invertible.

6.1. **The definition of the determinant.** Let A be a square $n \times n$ matrix. A *pattern* in A is a choice of n entries from A, so that one entry is chosen from each row and column. The product of the pattern is the product of chosen entries.

Example 6.1. The patterns in the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\begin{bmatrix} a & \\ & d \end{bmatrix}$ and $\begin{bmatrix} & b \\ c & \end{bmatrix}$.

Each pair of entries in the pattern is either oriented southwest-northeast (SW-NE) or southeast-northwest (SE-NW). The pair is said to be an *inversion* if it is oriented SW-NE. The *sign* of the pattern is $\operatorname{sign}(P) = (-1)^{\#inversions}$, that is 1 if the number of inversions is even, and -1 if the number of inversions is odd.

Example 6.2. ad is not inverted, bc is. The sign of ad is 1, the sign of bc is -1.

The determinant of A is defined by

$$\det(A) = \sum_{\text{patterns } P} (-1)^{\#\text{inversions}(P)} \text{product of entries}(P).$$

Example 6.3. The determinant of a 2×2 matrix is $\det(A) = ad - bc$.

We can ignore patterns that contain a zero, since these don't contribute to the determinant.

Example 6.4. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$. The non-zero patterns are $\begin{bmatrix} 1 & & \\ & 4 & \\ & & 7 \end{bmatrix}$ (no inversions) $\begin{bmatrix} & & 2 \\ & 4 & \\ 5 & & \end{bmatrix}$ (3 inversions) and $\begin{bmatrix} & & 2 \\ 3 & & \\ & 6 & \end{bmatrix}$ (2 inversions). So the determinant is

$$\det(A) = (1)(4)(7) + (-1)^3(5)(4)(2) + (-1)^2(3)(6)(2) = 24.$$

Therefore, the matrix is invertible.

Example 6.5. Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}.$$

The only non-zero choice from the first row is 1. The only non-zero choice from the second row, that is not in the same column as 1, is 3. In the same way, one sees that the only possible non-zero choices from the third and fourth rows are 6 and 10. There are no inversions in this pattern. Therefore, the determinant is

$$\det(A) = (1)(3)(6)(10) = 180$$

and the matrix is invertible. More generally, the same reasoning shows

Theorem 6.6. *Let A be upper triangular, lower triangular, or diagonal. Then the determinant is the product of diagonal entries. Therefore, A is invertible if non of the diagonal entries are zero.*

6.2. **Properties of the determinant.**

(1) (Transpose) Let A be a square matrix. Then $\det(A) = \det(A^T)$.

For every pattern in A flips over into a transpose for A^T, and vice-versa. For instance, (2)(3)(6) is a pattern in both

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}.$$

If a pair is oriented SW-NE before in A, the flipped pair is also oriented SW-NE:

$$\begin{bmatrix} & & 2 \\ 3 & & \\ & & \end{bmatrix}, \quad \begin{bmatrix} & & 3 \\ & & \\ 2 & & \end{bmatrix}$$

So the number of inversions in both patterns is the same. Since the determinant is the sum over patterns, with sign given by the number of inversions, this shows

(2) (Switching Two Rows) Let B equal the matrix A with two rows switched. Then $\det(B) = -\det(A)$.

Example 6.7.

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} = -\det \begin{bmatrix} 3 & 4 & 0 \\ 1 & 0 & 2 \\ 5 & 6 & 7 \end{bmatrix}.$$

Every pattern in B corresponds to a pattern in A but the number of inversions is different. Say row i is switched with row j. For every row k in between, the pair of entries in row i and row k switches from NE-SW to NW-SE or vice-versa. Similarly for the pair of entries in row j and row k. The pair of entries in rows i,j also switches from NE-SW to NW-SE. As a result the number of inversions changes by 2 times the number of rows in between, plus 1. So the sign $(-1)^{\#\text{inversions}}$ switches from $+$ to $-$, or vice-versa.

(3) (Equal Rows) If A has two rows equal, $\det(A) = 0$.

Example 6.8.

$$\det \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 0$$

because the first and third rows are equal.

Let B be the matrix with the two rows switched, that is, $B = A$. Then $\det(A) = -\det(B) = \det(A)$ which can only happen if $\det(A) = 0$.

(4) (Summing rows or columns) Let $\mathbf{v}, \mathbf{w}$ be n-vectors. Let A, B, C be square matrices so that A, B, C are all equal except that one of the rows is $\mathbf{v}$ for A, $\mathbf{w}$ for B, and $\mathbf{v}+\mathbf{w}$ for C. Then $\det(C) = \det(A) + \det(B)$.

Note it is *not* true that $\det(A+B) = \det(A) + \det(B)$; *it is only true if rows or columns are added*!

This is best proved later, using cofactor expansion.

(5) If B is the matrix obtained by multiplying row i by c, then $\det(B) = c\det(A)$.

Example 6.9.

$$\det\begin{bmatrix}1 & 0 & 2\\ 9 & 12 & 0\\ 5 & 6 & 7\end{bmatrix} = 3\det\begin{bmatrix}1 & 0 & 2\\ 3 & 4 & 0\\ 5 & 6 & 7\end{bmatrix}$$

because the second row has been multiplies by 3.

If B is obtained from A by multiplying every row by c, then $\det(B) = c^n \det(A)$. That is,

$$\det(cA) = c^n \det(A).$$

A common mistake is to forget the superscript n. In particular, $\det(-A)$ *is not equal to* $-\det(A)$ unless the size n is odd.

(6) (Adding one row to another) If B is obtained from A by adding a multiple of one row to another, then $\det(B) = \det(A)$.

This is a consequence of the previous two results: Say row i of A is $\mathbf{v}$, and row j is $\mathbf{w}$, and B has row i equal to $\mathbf{v} + c\mathbf{w}$. Then $\det(B) = \det(A) + c\det(C)$, where C is the matrix obtained by substituting $\mathbf{w}$ into row i. But then C has two rows equal, so $\det(C) = 0$.

Example 6.10. $\det\begin{bmatrix}1 & 0 & 2\\ 0 & 4 & -6\\ 5 & 6 & 7\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 2\\ 3 & 4 & 0\\ 5 & 6 & 7\end{bmatrix}$ because three times the first row has been subtracted from the second.

Now we're ready to show that the determinant determines whether the matrix is invertible.

Theorem 6.11. *The following are equivalent for a square matrix A:*

(1) $\det(A) \neq 0$
(2) $\mathrm{rref}(A) = I$
(3) *A is invertible.*

Proof. We already proved (2) $\iff$ (3). It remains to prove (1) $\iff$ (2).

Suppose the in the course of reducing A to its reduced row-echelon form, we do a number of row operations, such as adding multiples of rows to other rows, multiplying rows by non-zero numbers $c_1, \ldots, c_r$, and switching rows s times.

Under the first type of row operation, the determinant is unchanged. Under the second type, the determinant is multiplied by c_i. Under the third, the determinant changes sign.

$$\det(\mathrm{rref}(A)) = (-1)^s c_1 \ldots c_r \det(A).$$

This shows that $\det(A)$ is non-zero if and only if $\det(\mathrm{rref}(A))$ is non-zero.

The $\mathrm{rref}(A)$ is upper triangular, by (3) in its definition (4.3). So its determinant is non-zero if and only if there are non-zero numbers along the diagonal. If $\det(A)$ is non-zero, each of these must be a leading 1. But then the rref must equal the identity. □

Using similar techniques we can show the following.

Theorem 6.12. *For any square matrices A, B,* $\det(AB) = \det(A)\det(B)$.

Proof. Look at the vector equation $A\mathbf{x} = AB\mathbf{y}$. If A is invertible, this equation has solution $\mathbf{x} = B\mathbf{y}$. Suppose in the elimination s rows get switched, and rows get multiplied by non-zero numbers $c_1, \ldots, c_r$. Then
$$\det(A)(-1)^s c_1 \ldots c_n = \det(I) = 1.$$
Since the same operations happen on the right,
$$\det(AB)(-1)^s c_1 \ldots c_r = \det(B).$$
But the left-hand side is $1/\det(A)$. □

Corollary 6.13. (1) *If A is invertible then* $\det(A^{-1}) = 1/\det(A)$.
(2) *AB is invertible if and only if A is invertible and B is invertible.*
(3) *A^n is invertible if and only if A is invertible.*

Proof. (a) $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$. Now divide by $\det(A^{-1})$ on both sides. (a) $\det(AB) = \det(A)\det(B)$, so $\det(AB)$ is non-zero exactly if $\det(A)$ and $\det(B)$ are non-zero. (b) is similar. □

7. Cofactors

In this section we want to explain how the formula for 2×2 matrices
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
generalizes to bigger size.

7.1. **Cofactors.** Let A be a square $n \times n$ matrix. Let M_{ij} denote the matrix A with row i and column j deleted. M_{ij} is called the ij-th *minor* of A.

Example 7.1. The 13-th minor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is $M_{13} = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$.

The number
$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$
is the ij-th *cofactor* of A. The signs $(-1)^{i+j}$ are given by the *table of alternating signs*, for example, for 3×3 the table is
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

Example 7.2. The 13 cofactor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is
$$A_{13} = (-1)^{1+3}\det(\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}) = +(0-8) = -8.$$

Example 7.3. The cofactors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $d, -c, -b, a$.

Here is the reason we are interested in cofactors:

Theorem 7.4. *For any i, the dot product of the i-th row of A with the cofactors for the i-th row is* $\det(A)$. *If i is not equal to j, the dot product of the i-th row of A with the cofactors for the j-th row is equal to* 0. *That is,*

$$a_{i1}A_{i1} + \dots a_{in}A_{in} = \det(A), \qquad a_{i1}A_{j1} + \dots a_{jn}A_{jn} = 0. \tag{4}$$

Before we explain the theorem, here is an example.

Example 7.5. The cofactors for the first row of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ $+(28-0) = 28, -(0-12) = 12, +(0-8) = -8$. The dot product of $[28\ 12\ \ -8]$ with the first row of A is $28(1) + 12(3) - 8(5) = 24$. The dot product with the second row is $28(0) + 12(4) - 8(6) = 0$. The dot product with the third row is $28(2) + 12(0) - 8(7) = 0$.

Proof. Now we prove the theorem. Each pattern in A contains exactly one element in row i, say a_{ik}. The remaining chosen entries form a pattern in M_{ij}. So the product of entries appears in the sum (4). Conversely, any pattern in M_{ij} defines a pattern in A, by adding the entry a_{ij}. So the terms in the sum

$$a_{i1}\det(M_{i1}) + \dots a_{in}\det(M_{in})$$

are the same as those that appear in $\det(A)$.

It remains to explain the sign $(-1)^{i+j}$. The number of inversions in the pattern in A_{ij} is the number of inversions in the pattern in M_{ij}, plus the number v of inversions of pairs containing a_{ij}. Let's compute v. The matrix M_{ij} is naturally broken up into 4 parts: the entries that lie NE, NW, SE, SW of a_{ij}. We have

$$v = \#\text{NE entries} + \text{SW entries}.$$

Since there is only one chosen entries in each row and column

$$\#\text{NE entries} = (i-1) - \#\text{NW entries}, \quad \#\text{SW entries} = (j-1) - \#\text{NW entries}.$$

So

$$v = i + j - 2 - 2\#\text{NW entries}$$

which implies

$$(-1)^v = (-1)^{i+j}.$$

This proves the first part of (4).

Now suppose we take the dot product of row j in A with the cofactors for row i. Let B be the matrix obtained from A by replacing row i with row j. The cofactors for row i are the same for both B and A. The dot product of the j-th row of A with the cofactors, is the same as the dot product of the i-th row of B, with the cofactors. By the first part of (4), applied to B, the result is $\det(B)$. But since B has two rows equal, $\det(B) = 0$. $\square$

7.2. **Cofactor expansion of the determinant.** The first formula in (4) is called the *cofactor expansion* of the determinant along row i. For instance, suppose we want to compute the determinant of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}.$$

We choose a row along which to expand, say the second. We take each entry in the row, and multiply by the determinant of the corresponding minor, with the appropriate sign from the table of signs:

$$\det(A) = -0 \det \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix} + 4 \det \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix} - 6 \det \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = 24.$$

The same thing works for any column. For any i, the dot product of the i-th column of A with the cofactors for the j-th column is $\det(A)$, if $i = j$, and 0 otherwise.

Example 7.6. Suppose we want to find the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \end{bmatrix}.$$

There is only one non-zero entry in the fourth column; therefore it's best to expand along that column. We get

$$\det(A) = -0 + 4 \det \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} - 0 + 0.$$

Now there is only one non-zero entry in the third column, so we expand along it:

$$\det(A) = 4(0 - 0 + 7 \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 28(1(4) - 2(3)) = -56.$$

The *adjoint* of A is the transpose of the matrix of cofactors. That is, the ij-the entry of $\mathrm{adj}(A)$ is the ji-th cofactor A_{ji}.

Example 7.7. The adjoint of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is the matrix

$$\mathrm{adj}(A) = \begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}.$$

7.3. **The cofactor formula for the inverse.** Here is the promised formula for the inverse:

Theorem 7.8. $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$.

Proof. It suffices to show that $A \, \mathrm{adj}(A) = \det(A) I$. The dot product of the i-th row of A with the j-th column of $\mathrm{adj}(A)$ is the dot product of the i-th row of A with the cofactors for the j-th row. By (4), this equals $\det(A)$ if $i = j$, and 0 otherwise. These are the same as the entries of the matrix $\det(A) I$. $\square$

Example 7.9. The inverse of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{24} \begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}.$$

The cofactor formula is particularly useful when there are unknowns in the matrix.

Example 7.10. Find the inverse of $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$. Since A is upper triangular, the determinant is the product of diagonal entries $\det(A) = a^3$. The adjoint is

$$\operatorname{adj}(A) = \begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}.$$

The inverse is

$$A^{-1} = \frac{1}{a^3} \begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}.$$

Sometimes when the matrices contain unknowns it's easier to find the determinant using the row operations.

Example 7.11. Suppose we want to the unique polynomial passing through (x_1, y_1), (x_2, y_2) and (x_3, y_3). We write $f(x) = a + bx + cx^2$. The equations

$$a + bx_1 + cx_1^2 = y_1, \quad a + bx_2 + cx_2^2 = y_2, \quad a + bx_3 + cx_3^2 = y_3$$

can be written in matrix form

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

The determinant of the matrix is

$$\det(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix}$$

since subtracting the first row from the second and third does not change the determinant. Multiplying the second by $1/(x_2 - x_1)$ and subtracting $(x_3 - x_1)$ times the second row from the third gives

$$(x_2 - x_1) \det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & \frac{x_2^2 - x_1^2}{x_2 - x_1} \\ 0 & 0 & (x_3^2 - x_1^2) - \frac{(x_3 - x_1)(x_2^2 - x_1^2)}{x_2 - x_1} \end{bmatrix}.$$

So the determinant is

$$\begin{aligned}
&(x_2 - x_1)((x_3^2 - x_1^2) - \frac{(x_3 - x_1)(x_2^2 - x_1^2)}{x_2 - x_1}) \\
= \;& (x_2 - x_1)((x_3 - x_1)(x_3 + x_1) - (x_3 - x_1)(x_2 + x_1)) \\
= \;& (x_2 - x_1)((x_3 - x_1)(x_3 + x_1 - x_2 - x_1)) \\
= \;& (x_2 - x - 1)(x_3 - x_1)(x_3 - x_2).
\end{aligned}$$

This is called a *Vandermonde determinant.* You can easily guess how it generalizes to higher size.

8. Linear transformations

8.1. **Definition of a linear transformation.** A function from $\mathbb{R}$ to $\mathbb{R}$ assigns to any real number x another real number $f(x)$.

A *map* T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is similar, but it assigns to any vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, called the value of T at $\mathbf{x}$.

For example,

$$T_1[x_1\ x_2] = [5x_1 + 2x_2\ 3x_1 - x_2]$$
$$T_2[x_1\ x_2] = [x_1^2 - x_2^2\ x_1^2 + x_2^2]$$
$$T_3[x_1\ x_2] = [x_1 + 2\ x_2 - 3]$$

are all maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

A map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a *linear transformation* if it preserves vector addition and scalar multiplication, that is, if

(1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(2) $T(c\mathbf{x}) = cT(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$, $c \in \mathbb{R}$.

These conditions can be combined into a single condition, that T preserves linear combinations, that is,

$$T(\underset{\sim}{\mathbf{x}} + d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y}).$$

Example 8.1. Of the three maps $\mathbb{R}^2$ to $\mathbb{R}^2$ above, only the T_1 is a linear transformation. In fact, if A is the matrix $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$ then $T_1(\mathbf{x}) = A\mathbf{x}$. So

$$T(c\mathbf{x} + d\mathbf{y}) = A(c\mathbf{x} + d\mathbf{y}) = cA\mathbf{x} + dA\mathbf{y} = cT(\mathbf{x}) + dT(\mathbf{y}).$$

More generally, any map $T(\mathbf{x})$ of the form $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

The map T_2 fails because, for example,

$$T(2[3\ 0]) = T([6\ 0]) = [36\ 36]$$

but

$$2T([3\ 0]) = 2[9\ 9] = [18\ 18].$$

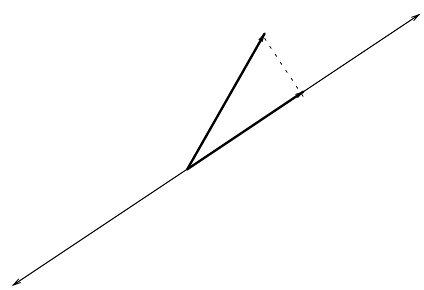
The map T_3 fails because, for example,

$$T_3[1\ 0] + T_3[2\ 0] = [3\ 0] + [4\ 0] = [7\ 0]$$

but $T_3[3\ 0] = [5\ 0.$

8.2. **Examples of linear transformations in two dimensions.** Let's look at some examples of linear transformations in $\mathbb{R}^2$.

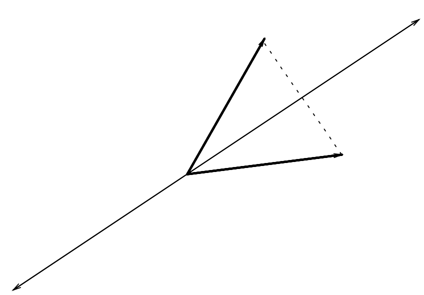
(1) Let L be a line in $\mathbb{R}^2$ passing through 0. For any $\mathbf{x} \in \mathbb{R}^2$, define $P(x)$ to be the vector whose head is the closest point to the head of $\mathbf{x}$ in L.



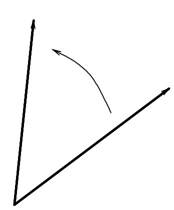
P is *orthogonal projection* onto L. Let's check graphically that it is a linear transformation:

(Figure)

(2) Let L be a line in $\mathbb{R}^2$ passing through 0. For any vector $\mathbf{x} \in \mathbb{R}^2$, define $S(x)$ to be the reflection of $\mathbf{x}$ over L. Then S is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.



(3) Let θ be an angle, and for any vector $\mathbf{x}$ let $R(x)$ be the rotation counterclockwise of $\mathbf{x}$ around 0 by angle θ. Then T is a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$.



8.3. **The main theorem about linear transformations.**

Theorem 8.2. *Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A, called the*matrix for the linear transformation.

Proof. Define $\mathbf{e}_1 = [1\ 0 \ldots 0], \mathbf{e}_2 = [0\ 1\ 0 \ldots 0], \ldots, \mathbf{e}_n = [0\ 0 \ldots 0\ 1]$. Define A to be the matrix whose columns are $A\mathbf{e}_1, \ldots, A\mathbf{e}_n$. For any vector $\mathbf{x}$ we have

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix} \\ &= x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n. \end{aligned}$$

Since T preserves linear combinations

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \ldots x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + \ldots x_n T(\mathbf{e}_n) \\ &= [T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)]\mathbf{x} \\ &= A\mathbf{x}. \end{aligned}$$

□

8.4. **Examples in two dimensions.** Let L be the line with slope 1 in $\mathbb{R}^2$, passing through 0. Let's find the matrix for projection onto L.

The closest vector to $\mathbf{e}_1$ in L is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. The closest vector to $\mathbf{e}_2$ is the same vector. Therefore,

$$P(\mathbf{x}) = A\mathbf{x}, \text{ where} A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Suppose we want to now find the closest vector to $[5\ 2]$ in L. We multiply by A to get

$$A \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}.$$

The matrix for reflection is similar. The reflection of $\mathbf{e}_1$ through L is $\mathbf{e}_2$, and the reflection of $\mathbf{e}_2$ through L is $\mathbf{e}_1$. So the matrix for reflection is

$$A = [\mathbf{e}_2\ \mathbf{e}_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now let R be rotation around 0 by angle θ. We have

$$R(\mathbf{e}_1) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad R(\mathbf{e}_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.$$

So the matrix for R is

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Which of these matrices are invertible? Let's compute their determinants.

$$\det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 1/4 - 1/4 = 0.$$

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 - 1 = -1.$$

$$\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1.$$

In general, reflections and rotations are invertible; projections are not. In fact, the inverse of a reflection is just the same reflection, since $S(S(\mathbf{x})) = \mathbf{x}$. The inverse of a rotation by θ is rotation by $-\theta$.

8.5. **The matrix of a composition is the matrix product.** If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ are maps, the composition is the map

$$T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p, \quad \mathbf{x} \to T_2(T_1(\mathbf{x})).$$

If T_1 and T_2 are linear, then so is $T_2 \circ T_1$:

$$T_2(T_1((c\mathbf{x} + d\mathbf{y})) = T_2(cT_1(\mathbf{x}) + dT_2(\mathbf{y})) = cT_2(T_1(\mathbf{x})) + dT_2(T_1(\mathbf{y})).$$

Proposition 8.3. *Let T_1 and T_2 be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ with matrices A_1 and A_2. Then the matrix for $T_2 \circ T_1$ is $A_2 A_1$.*

Example 8.4. Suppose that P is orthogonal projection onto a line L. Since $P(v)$ already has been projected, $P(P(v)) = v$, that is, $P \circ P = P$. If A is the matrix for P, then $A^2 = A$.

Example 8.5. Suppose S is reflection over a line L in $\mathbb{R}^2$. Then $S(S(v)) = v$, that is, the reflection reflects back to the original vector. If A is the matrix for S, then $A^2 = I$.

Example 8.6. Suppose R_θ is rotation by θ, and R_φ is rotation by φ. The composition is rotation by $\theta + \varphi$,

$$R_\theta \circ R_\varphi = R_{\theta+\varphi}.$$

We get

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} = \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix}.$$

The left hand side is

$$\begin{bmatrix} \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) & -\cos(\theta)\sin(\varphi) - \sin(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) \end{bmatrix}.$$

Equating the entries of the matrices we get the *angle-sum* formulas.

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

$$\sin(\theta + \varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi).$$

Example 8.7. Derive a formula for the cos of 3θ, using the same method.

9. Subspaces

Recall that $\mathbb{R}^n$ is the set of all n-vectors. A *subspace* V of $\mathbb{R}^n$ is a subset that staisfies the following three properties:

(1) V contains the zero vector 0.
(2) V is closed under vector addition: if v, w are in V then $v + w$ is also in V
(3) V is closed under scalar multiplication: if v is in V and c is a scalar then cv is also in V.

Properties (b) and (c) are equivalent to saying that V is closed under linear combination: if v, w are in V and c, d are scalars then $cv + dw$ is also in V.

Property (a) is equivalent to saying that V is non-empty. This is because if V contains at least one vector v, then it also has to contain $-v$, by (c) and so contain $v + (-v) = 0$, by (b). Usually, when we check that a subset V is a subspace, we will only verify properties (b) and (c), since (a) is usually obvious.

Example 9.1. The set V of all vectors of the form $[x\ x^2]$ is not a subspace, because it is not closed under scalar multiplication. For $v = [2\ 4]$ is in V, but $\frac{1}{2}v = [1\ 2]$ is not.

Example 9.2. The set V of all vectors of the form $[x\ 5x]$ is a subspace. In fact, if A is the matrix $[-5\ 1]$, then V is the set of vectors such that $Av = 0$.

More generally,

Definition 9.3. For any matrix A, the nullspace of A is the set of all vectors v such that $Av = 0$.

Lemma 9.4. *For any matrix A, the nullspace of A is a subspace.*

Proof. We check $V = \text{nullspace}(A)$ is closed under linear combinations: Assume that v, w are V. Then by definition $Av = Aw = 0$. This implies

$$A(cv + dw) = cAv + dAw = 0$$

so $cv + dw$ is also in V. □

Example 9.5. The set V of all vectors of the form $[x\ 2x + 1]$ is not a subspace, even though there are no higher order terms. V is closed under neither scalar multiplication nor vector addition; for instance, $[0\ 1]$ is in V, but twice it, $[0\ 2]$ is not.

Example 9.6. The set V of all vectors of the form $[x\ y]$ with $x, y \geq 0$ is closed under vector addition, but not scalar multiplication by negative numbers, so it is not a subspace.

Example 9.7. The set V of all vectors that are either in the x-axis or the y-axis, that is, $[x\ y]$ such that either x or y is zero, is closed under scalar multiplication but not vector additions, so it is not a subspace.

9.1. **Properties of Subspaces.**

Definition 9.8. If V and W are subspaces their *intersection* $V \cap W$ is the set of all vectors v that are in both V and in W. The *union* $V \cup W$ is the set of vectors that are in either V or in W. The *sum* $V + W$ is the set of vectors of the form $v + w$, for some v in V and w in W.

Example 9.9. If V is the x-axis and W is the y-axis in $\mathbb{R}^3$, then $V \cap W$ is just the origin, a single point; $V \cup W$ is the union of the two axis; $V + W$ is the xy-plane.

Theorem 9.10. (1) *Any subspace V must contain* 0.
(2) *The intersection $V \cap W$ of two subspaces V, W is a subspace.*
(3) *The union $V \cup W$ of two subspaces V, W is not in general a subspace.*
(4) *The sum $V + W$ of two subspaces V, W is a subspace.*

Proof. (1) Take any vector v in V and multiply by $c = 0$. Since V is closed under scalar multiplication, $cv = 0v = 0$ is also in V. (2) If v, w are in both V and W, then $cv + dw$ is in V and in W, and so in $V \cap W$. (3) See the example above. (4) An element in $V + W$ is of the form $v + w$ for some v, w. Any scalar multiple $c(v + w) = cv + cw$ is also of this form, so $V + W$is closed under scalar multiplication. We have to show that if we take two elements of $V + W$, they sum to another element. Suppose the second element is $v' + w'$, Then $v + w + (v' + w') = (v + v') + (w + w')$ which is also in $V + W$. □

10. Span and Linear independence

10.1. **Span.** Let $v_1, \dots, v_r$ be vectors in $\mathbb{R}^n$. The *span* of $v_1, \dots, v_r$ is the set of linear combinations

$$c_1 v_1 + \dots + c_r v_r.$$

Example 10.1. The span of a single non-zero vector v is the set of all cv, that is, the line through v.

Example 10.2. The span of the vectors $[1\ -1\ 0]$ and $[0\ -1\ 1]$ is the set of all combinations

$$a[1\ -1\ 0] + b[0\ -1\ 1] = [a\ -a - b\ b].$$

Any vector of this form has $x + y + z = 0$. Conversely, any vector with $x + y + z = 0$ can be written as

$$[x\ -x - y\ z] = a[1\ -1\ 0] + b[0\ 1\ -1]$$

where $a = x$ and $b = y$. So the span is the plan $x + y + z = 0$.

Proposition 10.3. *The span of any set of vectors is a subspace.*

Proof. Closed under $+$: $(c_1 v_1 + \dots c_r v_r) + (d_1 v_1 + \dots d_r v_r) = (c_1 + d_1)v_1 + \dots (c_r + d_r)v_r$. Closer under $\cdot$: $k(c_1 v_1 + \dots c_r v_r) = (kc_1)v_1 + \dots (kc_r)v_r$. □

Here is the algorithm for checking whether a set $v_1, \dots, v_r$ spans $\mathbb{R}^n$: Write the equation

$$c_1 v_1 + \dots c_n v_n = v$$

in matrix form. We want to know whether it always has a solution. This is equivalent to showing that the row-echelon form has no rows of zeros.

Example 10.4. Determine whether $[1\ -1\ 0], [1\ 0\ -1], [0\ 1\ -1]$ span $\mathbb{R}^3$.

Proposition 10.5. *If $v_1, \dots, v_r$ spans $\mathbb{R}^n$, then r must be at least n.*

10.2. **Linear independence.**

Definition 10.6. Vectors $v_1, \dots, v_r$ are *linearly independent* (or independent, for short) if no vector in the list is a combination of the others. If $v_1, \dots, v_r$ are not independent, they are *dependent.*

Example 10.7. Two vectors are independent if and only if they are not proportional. For example, $[-1\ 0\ 1]$ and $[-2\ 0\ 2]$ are dependent, because $[-2\ 0\ 2] = 2[-1\ 0\ 1]$. But $[-1\ 0\ 1]$ and $[2\ 0\ 2]$ are indepedent.

Example 10.8. Three vectors are independent of none of the vectors lies in the plane spanned by the other two. For example, $[1\ -1\ 0], [0\ -1\ 1], [1\ 0\ -1]$ is dependent, because $[1\ 0\ -1] = [1\ -1\ 0] - [0\ -1\ 1]$ lies the plane spanned by $[1\ -1\ 0], [0\ -1\ 1]$.

Here are some equivalent definitions:

Theorem 10.9. *Vectors $v_1, \ldots, v_r$ are dependent (that is, not independent) if and only if there is a subset that has the same span as $v_1, \ldots, v_r$.*

Definition 10.10. A dependence relation on $v_1, \ldots, v_r$ is a collection of scalars $c_1, \ldots, c_r$ not all zero such that $c_1 v_1 + \ldots c_r v_r = 0$.

Theorem 10.11. *Vectors $v_1, \ldots, v_r$ are independent if and only if there is no dependence relation on them.*

Example 10.12. For what values of c are the vectors $[0\ 0\ -1], [1\ 1\ 2], [1\ 1\ c]$ independent?

Example 10.13. For what values of c are the vectors $[0\ 1\ -1], [1\ 1\ 2], [1\ 1\ c]$ independent?

Here is the algorithm for checking whether a set of vectors $v_1, \ldots, v_r$ is linearly independent. Write the equation

$$c_1 v_1 + \ldots c_r v_r = 0$$

in matrix form. There are non-trivial solutions if and only if there is a column without a leading one. In this case, the vectors are dependent, since any non-trivial solution is a dependence relation.

11. Basis and Dimension

Definition 11.1. A set of vectors $v_1, \ldots, v_r$ is a basis for a vector space V if (1) $v_1, \ldots, v_r$ is linearly independent and (2) $v_1, \ldots, v_r$ spans V.

Example 11.2. $e_1 = [1\ 0\ 0 \ldots\ 0], e_2 = [0\ 1\ 0 \ldots\ 0], \ldots, e_n = [0\ 0\ ldots\ 0\ 1]$ is the *standard basis* for $\mathbb{R}^n$. Linear independence: no e_i is a combination of the others, since e_i has a 1 in the i-th entry and the other vectors have i-th entry 0. Span:

$$[x_1\ x_2\ \ldots\ x_n] = x_1[1\ 0\ \ldots\ 0] + x_2[0\ 1\ 0 \ldots\ 0] + \ldots + x_n[0\ 0\ ldots\ 0\ 1] = x_1 e_1 + \ldots x_n e_n.$$

The general procedure for finding a basis is the following: Find an expression for the general element of the vector space. Then, express it as a combination of linearly independent elements.

Example 11.3. Find a basis for the subspace $V = \{[a\ b\ c\ d],\ a = d,\ b = c\}$.

Example 11.4. Find a basis for the subspace $V = \{A \in M_{33}, A = A^T\}$.

Example 11.5. Find a basis for the subspace V of polynomials $p(x)$ of degree at most 6 such that $p(x) = p(-x)$.

Theorem 11.6. *A set $v_1, \ldots, v_r$ of vectors is a basis for a vector space V if and only if any vector in V can be written uniquely as a linear combination of these vectors: $v = c_1 v_1 + \ldots c_r v_r$ where $c_1, \ldots, c_r$ are unique.*

Theorem 11.7. *A set of vectors $v_1, \ldots, v_n$ are a basis for $\mathbb{R}^n$ if and only if the matrix A with columns $v_1, \ldots, v_n$ is invertible.*

Example 11.8. Show that if $v_1, \ldots, v_n$ is a basis, and A is an invertible matrix, that $Av_1, \ldots, Av_n$ is a also a basis.

Definition 11.9. A vector space is finite dimensional if it has a basis with a finite number of elements.

Theorem 11.10. *Any two bases $v_1, \dots, v_r$, $w_1, \dots, w_s$ for a finite dimensional vectors space V of $\mathbb{R}^n$ have the same number of elements.*

Definition 11.11. Let V be a vector space. The dimension of V is the number of elements in any basis.

Example 11.12. Find a basis for the space V of vectors perpendicular to $[1\ 1\ 1\ 1]$ and $[1\ 2\ 3\ 4]$.

12. Rank

The following three subspaces are associated to a matrix A.

Definition 12.1. The *nullspace* of A is the subspace of all vectors x such that $Ax = 0$, that is, the solution set to the homogeneous system corresponding to A. The *column space* of A is the span of the columns of A. The *row space* of A is the span of the rows of A.

12.1. The null-space: an example from epidemiology.

Let's look at the following model of flu epidemic. Suppose that in a population of 80 students, at any point in time there are w well students, s sick students, and i students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off. Find the matrix A that expresses the change $\Delta w, \Delta s, \Delta i$ in the numbers of well, sick, and immune students in terms of w, s, i.

$$\begin{bmatrix} \Delta w \\ \Delta s \\ \Delta \end{bmatrix} = A \begin{bmatrix} w \\ s \\ i \end{bmatrix}, \quad A = \begin{bmatrix} -.2 & 0 & 1 \\ +.2 & -.5 & 0 \\ 0 & +.5 & -1 \end{bmatrix}.$$

What is the practical meaning of the nullspace? It is the set of all vectors $[w\ s\ i]$ such that the change to the next week is zero. That is, the population stays the same. The vectors $[w\ s\ i]$ for which this happens are called *equilibrium* vectors. Let's find the null-space, by elimination.

$$\text{nullspace}(A) = \text{span} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

For instance, 50 well, 20 sick, and 10 immune is an equilibrium population.

Now back to mathematics.

Proposition 12.2. *Any vector in the null-space gives a depenendence relation on the columns of A.*

Proof. $Ax = 0$ means $x_1 v_1 + \dots x_n v_n = 0$. □

For instance, $[5\ 2\ 1]$ gives the relation .... So the last column is a combination of the first two.

12.2. The column space.

Now let's look at the column space. Since the last column is dependent, the span is the span of the first two. Multiplying by scalars does not change the span, so the span is the span of $[-1\ 1\ 0], [0\ -1\ 1]$. We already saw that this is the plan of vectors whose components sum to zero, in this case, $w + c + s = 0$.

Proposition 12.3. *The column space is the space of all vectors b for which $Ax = b$ has a solution.*

In this case, $Ax = b$ has a solution only if the components of b add up to zero. This is because only changes that preserve the total number 90 of students are possible, since we are assuming that no students die from the flu.

12.3. **Bases.** In this section, we call a column of A bound (resp. free) if the corresponding column in $\mathrm{ref}(A)$ contains (resp. does not contain) a leading 1.

Theorem 12.4. *Let A be any matrix.*

(1) *A basis for the null-space is obtained by solving the homogeneous system $Ax = 0$. The dimension of the null-space is the number of free variables.*
(2) *A basis for the column-space is given by the bound columns in A. The dimension of the column-space is the number of leading 1's.*
(3) *A basis for the row-space is given by the non-zero rows in the* $\mathrm{ref}(A)$.

Example 12.5. Find a basis for the nullspace, the row-space, and the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 7 & 9 & 11 \\ 3 & 6 & 10 & 13 & 16 \end{bmatrix}.$$

Proof. To find the null-space, we solve the homogeneous system $Ax = 0$. There is one basis vector for each free variable. The free columns can be expressed in terms of the bound columns, using the basis for the null-space. There are no dependence relations on the free columns. The non-zero rows □

12.4. **Rank.**

Definition 12.6. The *rank* of a matrix is the dimension of the column space, which is the same by the theorem above as the dimension of the row-space, and the same as the number of leading 1's. The *nullity* of a matrix is the dimension of the null-space, which by the theorem above is the same as the number of columns without leading 1's.

An $m \times n$ matrix has rank between 0 and the minimum of m and n, since there can be at most one leading 1 in each row and column.

Example 12.7. Find the rank and nullity of $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & 0 \\ 4 & 6 & 1 \end{bmatrix}$.

Theorem 12.8. *A matrix has rank 0 if and only if it is the zero matrix. A matrix A has rank n if and only if A is invertible.*

Proof. If there are no leading 1's, $\mathrm{ref}(A) = 0$, but then $A = 0$. If every column has a leading 1, $\mathrm{ref}(A) = I$, so A is invertible. □

Theorem 12.9. *(Rank-Nullity Theorem) The dimension of the column space (the rank) plus the dimension of the null-space (the nullity) is equal to the number of columns.*

For instance, if A is a 5×3 matrix with column space 2 dimensional, then the null-space is one-dimensional, so there are homogeneous solutions.

Corollary 12.10. *If A is an $m \times n$ matrix, and $m > n$ then the rows of A are dependent. If $n < m$ then the columns are dependent.*

Proof. If $m > n$ then the rank is at most n, so the dimension of the row-space is at most n. Since there are m vectors in an n-dimensional space, they are dependent. Similar for the case $n < m$ ☐

Theorem 12.11. *The rank of A is equal to the rank of A^T.*

Proof. The rank of A^T equals the dimension of the column-space of A^T equals the dimension of the row-space of A, which equals the dimension of the column-space of A, which equals the rank of A. ☐

12.5. Uniqueness of Reduced Row-Echelon Form.

Theorem 12.12. *The reduced row echelon form* $\mathrm{rref}(A)$ *of a matrix A is unique.*

Proof. Let W denote the row-space of A, $w_1, \ldots, w_r$ the non-zero rows of the $\mathrm{rref}(A)$ and i_j the column number of the leading 1 in w_j. Let $e_1, \ldots, e_n$ be the standard basis for R^n and

$$V_n = \mathrm{span}(e_n), V_{n-1} = \mathrm{span}(e_{n-1}, e_n), \ldots, V_1 = \mathrm{span}(e_1, \ldots, e_n).$$

By induction on $i = r - j + 1$, we show that $w_j, \ldots, w_r$ is the unique basis for the intersection $V_{i_j} \cap W$ such that the matrix with rows $w_j, \ldots, w_r$ is in reduced row-echelon form.

Case $i = 1$: Then $j = r$. w_r is the unique vector in the row-space in $V_{i_r} \cap W$ with leading coefficient 1.

Case i implies $i + 1$: Assume $w_{j+1}, \ldots, w_r$ is unique. Then there is a unique choice of w_j so that $w_j, \ldots, w_r$ is in reduced row-echelon form, since the entries above the leading 1's must be zero. ☐

13. Orthonormal bases/Orthogonal matrices

13.1. Orthogonality. Orthogonal is another name for perpendicular.

Definition 13.1. Vectors $v_1, \ldots, v_r$ are orthogonal if any two vectors v_i, v_j with $i \neq j$ are perpendicular, that is, $v_i \cdot v_j = 0$

Example 13.2. The standard basis $e_1, \ldots, e_n$ for R^n is orthogonal.

Example 13.3. $[1\ 0\ 0], [0\ 2\ 0], [0\ 0\ 3]$ are orthogonal.

Example 13.4. $[1\ 1], [1\ \ -1]$ are orthogonal.

Orthogonal vectors are particularly nice for a number of reasons. For instance,

Theorem 13.5. *Any orthogonal set of vectors $v_1, \ldots, v_r$ is linearly independent.*

Proof. First proof: Suppose one vector, say v_r, is a combination of the others:

$$v_r = c_1 v_1 + \ldots + c_{r-1} v_{r-1}.$$

Dot with v_r on both sides to get

$$v_r \cdot v_r = 0 \implies v_r = 0$$

which is a contradiction.

Second, more symmetric proof: Suppose that

$$c_1 v_1 + \ldots c_r v_r = 0.$$

Dot with v_1 to get

$$c_1 v_1 \cdot v_1 + c_1 v_2 \cdot v_1 + \ldots = c_1 v_1 \cdot v_1 = 0$$

which implies $c_1 = 0$. Dotting with v_2, v_3 etc. gives $c_2 = c_3 = 0$. So there are no dependence relations. ☐

A basis is orthogonal if it consists of orthogonal vectors.

Theorem 13.6. *Suppose that $v_1, \ldots, v_r$ is a basis for V, so that any v can be written uniquely*

$$v = c_1 v_1 + \ldots + c_r v_r.$$

If the basis is orthogonal, there is a simple expression for the coefficients $c_1, \ldots, c_r$:

$$c_1 = \frac{v \cdot v_j}{v_j \cdot v_j}.$$

Example 13.7. Suppose we want to express $[3\ 2]$ as a combination of $[1\ 1]$ and $[1\ -1]$. One way would be to solve the system

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 2 \end{array}\right].$$

But since $[1\ 1], [1\ -1]$ is an orthogonal basis, there is an easier way:

$$c_1 = \frac{[3\ 2] \cdot [1\ 1]}{[1\ 1] \cdot [1\ 1]} = \frac{5}{2}.$$

$$c_2 = \frac{[3\ 2] \cdot [1\ -1]}{[1\ -1] \cdot [1\ -1]} = \frac{1}{2}.$$

Example 13.8. Express $[3\ 2\ 1]$ as a combination of

$$v_1 = [1\ 1\ 1], v_2 = [1\ -1\ 0], v_3 = [1\ 1\ -2].$$

Step 1: check that v_1, v_2, v_3 forms an orthogonal basis. $v_1 \cdot v_2 = 1 - 1 = 0, v_2 \cdot v_3 = 1 - 1 = 0, v_1 \cdot v_3 = 1 + 1 - 2 = 0$. Step 2: Compute the coefficients c_1, c_2, c_3: $c_1 = (3 + 2 + 1)/(1 + 1 + 1) = 2, c_2 = (3 - 2)/(1 + 1), c_3 = (3 + 2 - 2)/(1 + 1 + 4) = 1/2$.

13.2. **Orthonormality.**

Definition 13.9. A set of vectors $v_1, \ldots, v_r$ is *orthonormal* if (1) $v_1, \ldots, v_r$ are orthogonal and (2) each of the vectors $v_1, \ldots, v_r$ is a unit vector.

Proposition 13.10. *Any orthogonal set of vectors $v_1, \ldots, v_r$ can be made into an orthonormal set by dividing by the lengths*

$$u_1 = \frac{v_1}{\|v_1\|}, \ldots, u_r = \frac{v_r}{\|v_r\|}.$$

Example 13.11. The vectors $[1\ 1\ 1], [1\ \ -1\ 0], [1\ 1\ \ -2]$ is orthogonal, but not orthonormal. To make it orthonormal we divide by the lengths to get

$$\frac{[1\ 1\ 1]}{\sqrt{3}},\ \frac{[1\ \ -1\ 0]}{\sqrt{2}},\ \frac{[1\ 1\ \ -2]}{\sqrt{6}}.$$

A basis $v_1, \ldots, v_r$ for a subspace V of R^n is called orthonormal if the vectors are orthonormal.

Proposition 13.12. *Suppose $u_1, \ldots, u_r$ is an orthonormal basis. Then any vector v can be written*

$$v = c_1 u_1 + \ldots + c_r u_r$$

where

$$c_j = v_j \cdot v.$$

Example 13.13. (Silly example) Let $e_1, \ldots, e_n$ be the standard basis for R^n. Then for any vector x, the formula gives $c_j = e_j \cdot x = x_j$ so that $x = x_1 e_1 + \ldots x_n e_n = [x_1\ 0\ 0 \ldots\ 0] + \ldots [0\ \ldots\ 0\ x_n] = [x_1\ \ldots\ x_n] = x$.

Example 13.14. Let's express $[3\ 2\ 1]$ in terms of $u_1 = \frac{[1\ 1\ 1]}{\sqrt{3}}$, $u_2 = \frac{[1\ -1\ 0]}{\sqrt{2}}$, $u_3 = \frac{[1\ 1\ -2]}{\sqrt{6}}$. We get

$$c_1 = 6/\sqrt{3}, \quad c_2 = 1/\sqrt{2},\ c_3 = 3/\sqrt{6}.$$

13.3. **Orthogonal Matrices.** The definition of an orthonormal basis can be written in matrix form. Let Q be the matrix with columns $v_1, \ldots, v_r$. Note that the rows of Q^T are $v_1, \ldots, v_r$. So $Q^T Q$ is the matrix whose entries are $v_i \cdot v_j$, that is, the rows of Q^T dotted with the columns of Q.

The following conditions are equivalent:

(1) $v_1, \ldots, v_r$ is orthonormal;
(2) $v_i \cdot v_j = 1$, if $i = j$, and 0 otherwise;
(3) The matrix whose entries are $v_i \cdot v_j$ is the identity matrix ;
(4) $Q^T Q = I$.

This motivates the following definition:

Definition 13.15. A matrix Q is orthogonal if and only if (1) Q is square and (2) $Q^T Q = I$.Equivalently, Q is orthogonal iff Q is square and $Q^{-1} = Q^T$. Equivalently, Q is orthogonal iff its columns form an orthonormal basis for R^n, where n is the number of columns of Q.

This explains why the inverse has some of the same properties as the transpose: the two operations are the same for a large class of matrices.

Example 13.16. The identity matrix I is orthogonal. Indeed, $I^{-1} = I = I^T$. The columns of I form the standard basis for R^n, which is orthonormal.

Orthogonal matrices have a number of nice properties:

Proposition 13.17. (1) *If Q is orthogonal, then so is Q^{-1}.*
(2) *If Q_1 and Q_2 are orthgonal, then so is $Q_1 Q_2$.*
(3) *If Q is orthogonal, then* $\det(Q) = \pm 1$.

Proof. (1) $Q^T = Q^{-1}$ implies $(Q^{-1})^T = (Q^T)^{-1} = (Q^{-1})^{-1}$. (2) is left to you. (3) If $Q^T Q = I$ then applying det to both sides we get

$$1 = \det(I) = \det(Q^T Q) = \det(Q^T)\det(Q) = \det(Q)^2$$

so $\det(Q) = \pm 1$. □

Example 13.18. Classify orthonormal bases for R^2. The first vector u_1 can be any unit vector. This means $u_1 = [\cos(\theta)\ \ \sin(\theta)]$ for some angle θ. The vector u_2 must be a unit vector perpendicular to u_1. There are only two possibilities: $u_2 = \pm[-\sin(\theta)\ \ \cos(\theta)]$.

Classify orthogonal 2×2 matrices. By what we have just said, the only possibilities are

$$Q_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad Q'_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & +\cos(\theta) \end{bmatrix}.$$

The matrix Q_θ has determinant $\cos^2(\theta) + \sin^2(\theta) = 1$; it is the matrix for the linear transformation given by counter-clockwise rotation by angle θ. The matrix Q'_θ has determinant $-\cos^2(\theta) - \sin^2(\theta) = -1$.

13.4. **Gram-Schmidt.** Any basis can be made into an orthonormal basis, by a procedure call the *Gram-Schmidt* process. Let's start with just two vectors. We define u_1 by making v_1 into a unit vector:

$$u_1 = v_1/\mathit{length}.$$

We want to define u_2 to be a unit vector perpendicular to u_1. It's easier to first construct a vector perpendicular to u_1, and then make it a unit vector, since changing the length doesn't change any angles. Let's try

$$v_2' = v_2 - cu_1.$$

In order to get $u_1 \cdot u_2 = 0$, we need

$$(v_2 - cu_1) \cdot u_1 = 0 \implies v_2 \cdot u_1 = cu_1 \cdot u_1 \implies c = v_2 \cdot u_1.$$

Hence

$$v_2' = (v_2 - (v_2 \cdot u_1)u_1), \quad u_2 = v_2/\|v_2\|.$$

Example 13.19. Make the vector $[3\ 2], [2\ 3]$ into an orthonormal basis using Gram-Schmidt.

$$u_1 = [3\ 2]/\sqrt{13}.$$

$$v_2' = [2\ 3] - \frac{12}{13}[3\ 2] = [-10/13\ 15/13].$$

$$u_2 = v_2'/\|v_2'\| = [-2\ 3]/\sqrt{13}.$$

Example 13.20. Make the basis [1 1 0], [0 1 1], [1 0 1] into an orthonormal basis using Gram-Schmidt.

$$u_1 = v_1/\|v_1\| = [1\ 1\ 0]/\sqrt{2}.$$

$$\begin{aligned} v_2' &= v_2 - (v_2 \cdot u_1)u_1) \\ &= ([0\ 1\ 1] - \frac{1}{2}[1\ 1\ 0]) \\ &= [-1/2\ 1/2\ 1] \end{aligned}$$

$$u_2 = v_2'/\|v_2'\| = [-11\ 2]/\sqrt{6}$$

$$\begin{aligned} v_3' &= v_3 - (v_3 \cdot u_1) \cdot u_1 - (v_3 \cdot u_2)u_2 \\ &= [1\ 0\ 1] - \frac{1}{2}[1\ 1\ 0] - \frac{1}{6}[-1\ 1\ 2] \\ &= [2/3\ \ -2/3\ 2/3] \end{aligned}$$

$$u_3 = v_3/\|v_3'\| = [1\ \ -1\ 1]/\sqrt{3}.$$

Theorem 13.21. *Let $v_1, \ldots, v_r$ be linearly independent. Then the formulas*

$$u_1 = v_1/\|v_1\|$$

$$v_2' = v_2 - (v_2 \cdot u_1)u_1, \quad u_2 = v_2'/\|v_2'\|, \quad \ldots$$

$$v_r' = v_r - (v_r \cdot u_1)u_1 - \ldots - (v_r \cdot u_{r-1})u_{r-1}, \quad u_r = v_r/\|v_r'\|$$

define an orthonormal basis for the span of $v_1, \ldots, v_r$.

Proof. By induction on r. Step $r = 1$: Clearly u_1 is a unit vector, with the same span as v_1. Step $r - 1 \implies r$. Suppose we have shown that $u_1, \ldots, u_{r-1}$ are orthonormal with the same span as $v_1, \ldots, v_{r-1}$. Since v_r is not a combination of $v_1, \ldots, v_{r-1}$,

$$v_r' = (v_r - (v_r \cdot u_1)u_1 - \ldots - (v_r \cdot u_{r-1})u_{r-1})$$

is non-zero. So u_r is also non-zero. Therefore, the formula makes sense, and it clearly defines a unit vector. It remains to check $u_r \cdot u_j = 0, j < r$. This follows from the formula above, since

$$v_r' \cdot u_j = (v_r \cdot u_j - 0 - \ldots - (v_r \cdot u_j)u_j \cdot u_j - 0 \ldots) = v_r \cdot u_j - v_r \cdot u_j = 0.$$

□

14. Orthogonal Projections

Definition 14.1. Let V be a subspace of R^n. The *orthogonal complement* $V^\perp$ of V is the set of all vectors w such that w is perpendicular to v for all vectors v in V. Equivalently, $w \cdot v$ for all v in V.

Note that

Lemma 14.2. *If $v_1, \ldots, v_r$ is a basis for V, then w is in $V^\perp$ if and only if w is perpendicular to $v_1, \ldots, v_r$.*

Proof. $wv_j = 0$ for $j = 1, \ldots, r$ implies $w(c_1v_1 + \ldots c_rv_r) = 0$ for any scalars c_j, which implies $wv = 0$ for all v in V. □

Example 14.3. Let $v = [1\ 2\ 3]$ in R^3 and let V be the span of V. The orthogonal complement is the set of all vectors perpendicular to v, that is the set of $w = [x\ y\ z]$ such that $x + 2y + 3z = 0$. $V^\perp$ is the plane perpendicular to (or with normal equal to) $[1\ 2\ 3]$.

Example 14.4. Let V be the span of $v_1 = [1\ 2\ 3]$ and $v_2 = [3\ 2\ 1]$. Then V is a plane and $V^\perp$ is the perpendicular to this plane, and so is a line. To compute the equation for $V^\perp$, we do elimination:

$$\begin{aligned} V^\perp &= \{[x\ y\ z],\ x + 2y + 3z = 0\} \\ &= \{[-2y - 3z\ y\ z]\} \\ &= \operatorname{span}[-2\ 1\ 0], [-3\ 0\ 1]. \end{aligned}$$

Note we have been using the null-space algorithm to find a basis for $V^\perp$. We can always do this because of the following:

Proposition 14.5. *For any matrix A, the nullspace of A is the orthogonal complement of the rowspace of A.*

Proof. w is in the null-space of A if and only if $Aw = 0$ if and only if each row v_j dotted with w gives 0. □

14.1. Properties of Orthogonal Complements.

Theorem 14.6.
1. *V and $V^\perp$ intersect in the zero vector.*
2. *If V has dimension r, then $V^\perp$ is a subspace of dimension $n - r$.*
3. *Any vector u in $\mathbb{R}^n$ may be written uniquely as a combination of a vector v in V and a vector w in $V^\perp$.*
4. *For any subspace V, $(V^\perp)^\perp = V$.*

Proof. (1) If u is in V and u is in $V^\perp$, then $u \cdot u = 0$, so $u = 0$.

(2) Since $v_1, \ldots, v_r$ are linearly independent, there is a leading one in every row. So there are r leading 1's. Therefore, $\dim V^\perp$ is the number of free variables, which is the number of columns without leading ones, which is $n - r$.

(3) Pick a basis $v_1, \ldots, v_r$ for V, and a basis $w_1, \ldots, w_{n-r}$ for $V^\perp$. Then $v_1, \ldots, v_r, w_1, \ldots, w_{n-r}$ is orthonormal, so linearly independent, so a basis for R^n. Hence any vector can be written uniquely

$$u = c_1 v_1 + \ldots + c_n w_{n-r}.$$

Let $v = c_1 v_1 + \ldots + c_r v_r$ and $w = c_{r+1} w_1 + \ldots c_n w_{n-r}$. Then v is in V and w is in W.

We prove that v and w are unique. Suppose $u = v' + w'$ with $v' \in V, w' \in W$. Then

$$v + w = v' + w' \implies v - v' = w' - w.$$

So $v - v' \in W$ and $w' - w \in V$. But this is a contradiction, by (1).

(4) $(V^\perp)^\perp$ is the set of vectors u such that u is perpendicular to any vector in $V^\perp$. Given any such vector, we may write it $u = v + w$ by (3). But then u is perpendicular to w so $u \cdot w = 0 + w \cdot w = 0$ which implies $w = 0$. Hence u is in V. Conversely, any vector v in V is perpendicular to $V^\perp$, and so lies in $(V^\perp)^\perp$. We have shown that V is contained in $(V^\perp)^\perp$ and vice-versa, so the two subspaces must be equal. □

14.2. Orthogonal Projections.

Definition 14.7. Let V be a subspace of R^n, and u a vector, and $u = v + w$ the decomposition given by (3) above. The *orthogonal projection* of u onto V is the vector v.

Example 14.8. Suppose V is the xy-plane and $u = [1\ 2\ 3]$. Then $V^\perp$ is the z-axis and the decomposition of u is

$$[1\ 2\ 3] = [1\ 2\ 0] + [0\ 0\ 3].$$

So $v = [1\ 2\ 0]$ is the projection of u onto V and $w = [0\ 0\ 3]$ is the projection of u onto $V^\perp$.

Theorem 14.9. *Let V be the span of a single vector v_1. Then the projection of u onto V is $v = \frac{u \cdot v_1}{v_1 \cdot v_1} v$ and the projection of u onto $V^\perp$ is $w = v - \frac{u \cdot v_1}{v_1 \cdot v_1}$.*

Proof. We write

$$u = cv_1 + (u - cv_1)$$

and solve for c so that

$$cv_1 \cdot (u - cv_1) = 0.$$

We get

$$cv_1 \cdot u = c^2 v_1 \cdot v_1 \implies c = \frac{v_1 \cdot u}{v_1 \cdot v_1}.$$

□

Example 14.10. Find the projection of the vector $u = [1\ 0\ 0]$ onto the span V of $v_1 = [1\ 2\ 3]$. Find the projection of u onto $V^\perp$.

Theorem 14.11. *Suppose that V is a subspace with orthogonal basis $v_1, \ldots, v_r$. Then the projection of u onto V is*

$$v = (u \cdot v_1)v_1 + \ldots (u \cdot v_r)v_r$$

and the projection of u onto $V^\perp$ is

$$w = v - (u \cdot v_1)v_1 + \ldots (u \cdot v_r)v_r.$$

Proof. We write

$$v = c_1 v_1 + \ldots c_r v_r, \quad w = u - c_1 v_1 + \ldots c_r v_r$$

and solve for $c_1, \ldots, c_r$ so that $w \cdot v_j = 0$ for $j = 1, \ldots, r$. □

Example 14.12. Find the projection of the vector $\mathbf{u} = [1\ 2\ 3]$ onto the subspace V spanned by $v_1 = [1\ 1\ 0]$ and $v_2 = [0\ 1\ 1]$.

14.3. Projection Matrices.

Theorem 14.13. *The map T that sends u to its projection v is a linear transformation. If $v_1, \ldots, v_r$ is an orthonormal basis, the matrix P for T is*

$$P = v_1 v_1^T + \ldots + v_r v_r^T.$$

If $v_1, \ldots, v_r$ is an orthogonal basis, the formula for the matrix T is

$$P = \frac{v_1 v_1^T}{v_1^T v_1} + \ldots + \frac{v_r v_r^T}{v_r^T v_r}.$$

If $v_1, \ldots, v_r$ is an arbitrary basis, the formula for the matrix is

$$P = A(A^T A)^{-1} A^T$$

where A is the matrix with columns $v_1, \dots, v_r$.

Proof. T is a linear transformation:

$T(v) = Pv$:

In the case $v_1, \dots, v_r$ is an arbitrary basis for V, Pu is the unique point in V such that Pu is a combination of $v_1, \dots, v_r$, and $Pu \cdot v_j = u \cdot v_j$ for each v_j. In matrix form, this means that

$$A^T P u = A^T u.$$

If $P = A(A^TA)^{-1}A^T$ then Pu is A times something, and so a combination of v_i's. Also $A^TPu = A^TA(A^TA)^{-1}A^Tu = A^Tu$ so $A^TP = A^T$. This shows the formula. □

Example 14.14. Find the matrix for projection onto the xy-plane.

Example 14.15. Find the matrix for projection onto the span of [1 1 0] and [0 1 1].

15. Least Squares Approximation

Suppose we want to find the line that best fits the data points $(0,0),(1,0), and (2,3)$. Before we saw how to set this problem up as a system of linear equations: We write $f(x) = c_1x + c_0$ and solve for a, b

$$c_1(0) + c_0 = 0$$
$$c_1(1) + c_0 = 0$$
$$c_1(2) + c_0 = 3.$$

Since the three points are not colinear, there is no solution. The problem is that the vector $b = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is not in the column space of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$. To fix this problem, we *project* the vector b onto the column space of A. This gives a vector Pb which is a close to b as possible, yet now has a solution. The equation

$$Ax = Pb$$

is called the least square equation. Since $b - Pb$ is in the perp of the column space,

$$A^T(b - Pb) = 0.$$

Hence $A^T(Ax - Pb) = 0$ which implies that

$$A^TAx = A^Tb.$$

Any solution is called a least square solution.

Example 15.1. In our case, this equation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

The solution is

$$c_1 = 3/2, \quad c_0 = -1/2.$$

Example 15.2. Apply the least squares method to find the closest line(s) to the data points $(0,0), (2,0), (2,3)$.

Example 15.3. Apply the least squares method to find the curve of the form $c_0 + c_1 x + c_2 x^2$ best fitting the points $(-1,1), (0,0), (1,1), (2,1)$.

Example 15.4. Find all the functions of the form $f(t) = c_0 + c_1 \cos(\pi t) + c_2 \cos(2\pi t)$. that are best fits for the data points $(-1/2, 1), (0,0), (1/2, 0)$.

16. Eigenvectors and eigenvalues

Consider the following mathematical model for the market for cola. Suppose $c(t)$ (resp. $p(t)$) is the number of Coke (resp. Pepsi) drinkers at time t months. Suppose each month, 10 percent of the Coke drinkers switch to become Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as t goes to infinity?

t	p	c
0	0	100
1	20	80
2	34	66

To set this up as a linear algebra problem we write

$$c(t+1) = .9c(t) + .2p(t)$$
$$p(t+1) = .1c(t) + .8p(t)$$

or in matrix form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) \text{ where } A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

This implies that

$$\mathbf{x}(t) = A\mathbf{x}(t-1) = A^2\mathbf{x}(t-2) = \ldots = A^t\mathbf{x}(0)$$

for any time t. The best method for solving this for large t is *eigenvectors/eigenvalues.*

Definition 16.1. An *eigenvector* of a square matrix A is a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some number λ, called the eigenvalue of $\mathbf{x}$. An *eigenvalue* of a square matrix A is a number λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, called an eigenvector for λ.

Geometrically, an eigenvector is a vector $\mathbf{x}$ such that $A\mathbf{x}$ lies in the same direction (or opposite direction) as the original vector. The eigenvalue λ is the "stretch factor".

Example 16.2. Say $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$ as above. Then $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $.7$. Also $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

Let's use these eigenvectors to solve the coke/pepsi problem described above. To begin, we write the *initial state vector* $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$ in terms of the eigenvectors:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 100 \end{bmatrix} = (200/3) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3) \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then

$$\mathbf{x}_t = \ldots A^t \mathbf{x}_0 = (200/3) A^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3) A^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (200/3)(.7)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (100/3)(1)^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

For t very large, $(.7)^t$ is approximately zero. So

$$\mathbf{x}_t \cong (100/3) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 66.7 \\ 33.3 \end{bmatrix}.$$

That is, in the long run 2/3 of the customers are with Coke, and 1/3 with Pepsi.

16.1. **Finding eigenvalues.** First we find the eigenvalues. The following are equivalent:

(1) λ is an eigenvalue of A.
(2) $A\mathbf{v} = \lambda \mathbf{v}$ for some vector $\mathbf{v} \neq 0$.
(3) $(A - \lambda I)\mathbf{v} = 0$ for some vector $\mathbf{v} \neq 0$.
(4) $\text{nullspace}(A - \lambda I) \neq 0$.
(5) $A - \lambda I$ is not invertible.
(6) $\det(A - \lambda I) = 0$.

So to find the eigenvalues we have to solve $\det(A - \lambda I) = 0$. The polynomial $\det(A - \lambda I)$ is the *characteristic polynomial* of A.

Example 16.3. If $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$ then

$$0 = \det(A - \lambda I) = \det(\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = (.9 - \lambda)(.8 - \lambda) - .02 = .7 - 1.7\lambda + \lambda^2.$$

Solving such an equation is equivalent to factoring

$$.7 - 1.7\lambda + \lambda^2 = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

We want to find numbers λ_1, λ_2 so that $\lambda_1 + \lambda_2 = 1.7$ and $\lambda_1 \lambda_2 = .7$. The solution is

$$\lambda_1 = 1, \quad \lambda_2 = .7.$$

This gives the eigenvalues above.

Example 16.4. Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. To find the eigenvalues we set

$$0 = \det(A - \lambda I) = \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}.$$

By expanding along the first row this equals

$$(2-\lambda)((2-\lambda)^2-1)-(-1)(-1)(2-\lambda)=(2-\lambda)(\lambda^2-4\lambda+3)-(2-\lambda)$$

$$=(2-\lambda)(\lambda^2-4\lambda+2)=(2-\lambda)(\lambda-(2+\sqrt{2}))(\lambda-(2-\sqrt{2})).$$

So the eigenvalues are

$$\lambda_1=2, \lambda_2=(2+\sqrt{2}), \lambda_3=(2-\sqrt{2}).$$

Example 16.5. Let $A=\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$. To find the eigenvalues we set

$$0=\det(A-\lambda I)=(1-\lambda)(1-\lambda)(1-\lambda).$$

So the eigenvalues are $\lambda=1,1,1$.

More generally, if A is upper or lower triangular or diagonal, then $A-\lambda I$ is also upper or lower triangular, so that $\det(A-\lambda I)$ is the product $(a_{11}-\lambda)\ldots(a_{nn}-\lambda)$. This shows

Theorem 16.6. *If A is upper or lower triangular or diagonal then the eigenvalues of A are the diagonal entries $a_{11},\ldots,a_{nn}$.*

16.2. **Finding eigenvectors.** Once we have found the eigenvalues, we can find the eigenvectors. The following are equivalent:

(1) v is an eigenvector of A with eigenvalues λ;
(2) $Av=\lambda v$
(3) $(A-\lambda I)v=0$
(4) v is in the nullspace of $A-\lambda I$.

So to find the eigenvectors we have to find the nullspace of $A-\lambda I$, for each eigenvalue λ.

Example 16.7. Let $A=\begin{bmatrix}.9&.2\\.1&.8\end{bmatrix}$. The eigenvalues are $\lambda=.7$ and $\lambda=1$. We compute

$$\begin{aligned}\text{nullspace}\, A-.7I &= \text{nullspace}\begin{bmatrix}.9&.2\\.1&.8\end{bmatrix}-\begin{bmatrix}.7&0\\0&.7\end{bmatrix}\\ &= \text{nullspace}\begin{bmatrix}.2&.2\\.1&.1\end{bmatrix}\\ &= \text{nullspace}\begin{bmatrix}1&1\\0&0\end{bmatrix}\\ &= \text{span}\begin{bmatrix}1\\-1\end{bmatrix}.\end{aligned}$$

$$\begin{aligned}
\text{nullspace } A-(1)I &= \text{nullspace} \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \text{nullspace} \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \\
&= \text{nullspace} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \\
&= \text{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\end{aligned}$$

So "the" eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

as we claimed above. Note these vectors are not unique: any multiples of $\mathbf{v}_1, \mathbf{v}_2$ are also eigenvectors.

Example 16.8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ so the eigenvalues are $1, 1, 3$. Then

$$\begin{aligned}
\text{nullspace } A-(1)I &= \text{nullspace} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \text{nullspace} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
\text{nullspace } A-(3)I &= \text{nullspace} \begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \text{nullspace} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 3/2 \\ 3/2 \\ 1 \end{bmatrix}.
\end{aligned}$$

So in this case there are two eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \begin{bmatrix} 3/2 \\ 3/2 \\ 1 \end{bmatrix}.$$

Example 16.9. Every vector is an eigenvector for the identity matrix I with eigenvalue 1.

16.3. **Properties of the eigenvalues.** The characteristic polynomial $\det(A - \lambda I)$ has degree n, so there are at most n solutions to $\det(A - \lambda I) = 0$. If there are exactly n solutions $\lambda_1, \ldots, \lambda_n$ to the equation $\det(A - \lambda I) = 0$ in the real numbers, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda),$$

we say that the eigenvalues of A are *all real*. This terminology will be explained later; for the moment we assume that all eigenvalues are real.

The number of times a factor $(\lambda_i - \lambda)$ appears is the *algebraic multiplicity* of λ_i.

Example 16.10. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Then $\det(A - \lambda I) = (1-\lambda)^2(3-\lambda)$ so $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue with algebraic multiplcity 1.

Theorem 16.11. *The number of real eigenvalues, counted with algebraic multiplicity, is at most n. The determinant of A is the product of the eigenvalues taken with algebraic multiplicity.*

Proof. We plug $\lambda = 0$ into the characteristic polynomial to get $\det(A) = (\lambda_1)(\lambda_2)\dots(\lambda_n)$. $\square$

Corollary 16.12. *A matrix is invertible only if 0 is not an eigenvalue.*

Proof. A is invertible, iff $\det(A) \neq 0$, iff none of the λ_i's is zero. $\square$

There is one other coefficient of the characteristic polynomial which has a simple interpretation. The *trace* of a square matrix A is the sum of the diagonal entries:

$$\mathrm{Tr}(A) = a_{11} + \ldots + a_{nn}.$$

Theorem 16.13. *The coefficient of $(-1)^{n+1}\lambda^{n-1}$ in the characteristic polynomial $\det(A - \lambda I)$ is the trace $\mathrm{Tr}(A)$.*

Proof. We expand to get

$$(\lambda_1 - \lambda)\dots(\lambda_n - \lambda) = (-1)^n\lambda^n + (\lambda_1 + \ldots + \lambda_n)(-1)^{n-1} + O(\lambda^{n-2})$$

where $O(\lambda^{n-2})$ means terms of order at most $n-2$ in λ. On the other hand, the only term in $\det(A - \lambda I)$ involving at least $n-1$ λ's is

$$(a_{11} - \lambda)\dots(a_{nn} - \lambda) = (-1)^n\lambda^n + (a_{11} + \ldots + a_{nn})(-1)^{n-1}\lambda^{n-1} + O(\lambda^{n-2}).$$

Equation the coefficients of λ^{n-1} finishes the proof. $\square$

Theorem 16.14. *The tranpose A^T of a square matrix A has the same eigenvalues as A.*

Proof. The characteristic polynomial

$$\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \lambda((A - \lambda I)^T) = \det(A - \lambda I).$$

So the eigenvalues, which are the roots of the characteristic polynomial, are also the same. $\square$

Theorem 16.15. *Suppose that the columns of A sum up to 1. Then $\lambda = 1$ is an eigenvalue for A.*

Proof. If the columns of A sum up to 1 then

$$[1\ 1\ 1 \ldots 1]\, A = [1\ 1\ 1 \ldots 1]\, A$$

which implies

$$A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

which implies that $\lambda = 1$ is an eigenvalue of A^T. By the Theorem, this implies implies that $\lambda = 1$ is an eigenvalue of A. $\square$

Suppose A is a matrix which represents a physical system in which the total number is preserved, e.g. the matrix in the Coke/Pepsi example

$$A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}.$$

Any such matrix has $\lambda = 1$ as an eigenvalue. This means that there is a vector v that is an equilibrium for the system, that is $Av = v$.

16.4. **Properties of the eigenvectors.** For any eigenvalue λ define

$$E_\lambda = \text{nullspace}(A - \lambda I).$$

This is the λ-*eigenspace for* A. The dimension of E_λ is called the *geometric multiplicity* of λ.

Theorem 16.16. *If $v_1, \ldots, v_r$ is a collection of vectors from different eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_r}$ then $v_1, \ldots, v_r$ are linearly independent.*

Proof. Suppose one, say v_r is a combination of the others

$$v_r = c_1 v_1 + \ldots c_{r-1} v_{r-1}.$$

Applying A to both sides we get

$$\lambda_r v_r = c_1 \lambda_1 v_1 + \ldots + c_{r-1} \lambda_{r-1} v_{r-1}.$$

Subtracting λ_r times the first equation we get

$$0 = c_1(\lambda_1 - \lambda_r) v_1 + \ldots + c_{r-1}(\lambda_{r-1} - \lambda_r) v_{r-1}.$$

By the inductive hypothesis, $v_1, \ldots, v_{r-1}$ are independent so

$$c_1(\lambda_1 - \lambda_r) = \ldots = c_{r-1}(\lambda_{r-1} - \lambda_r) = 0.$$

Since all the eigenvalues $\lambda_1, \ldots, \lambda_r$ are distinct, this implies that

$$c_1 = \ldots = c_{r-1} = 0$$

which shows that $v_1, \ldots, v_r$ are independent. □

17. Diagonalization

If an $n \times n$ matrix A has n independent eigenvectors $v_1, \ldots, v_n$, A is called *diagonalizable.* In this case the eigenvectors $v_1, \ldots, v_n$ form a basis for R^n called an *eigenbasis.*

Theorem 17.1. *If A is diagonalizable, then $A = SDS^{-1}$ where S is the matrix whose columns are the eigenvectors of A, and D is the diagonal matrix of eigenvalues.*

Example 17.2. Suppose $A == \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. Then

$$S = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} .7 & 0 \\ 0 & 1 \end{bmatrix}$$

then $A = SDS^{-1}$.

Example 17.3. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ with eigenvalues $1, 1, 3$ is *not* diagonalizable because there are only *two* independent eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3/2 \\ 3/2 \\ 1 \end{bmatrix}$.

Example 17.4. The identity matrix, or more generally any diagonal matrix is diagonalizable since it is already diagonal! In this case $S = I$ and $D = A$.

Finding the matrices S and D is called *diagonalizing* A.

Theorem 17.5. *If a matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then A is diagonalizable.*

Proof. Choose an eigenvector v_i for each eigenvalue λ_i. Since there are n-eigenvalues, there are n eigenvectors, independent by the theorem above. □

17.1. **Application to matrix powers.** Suppose A is a square matrix, and we want to find a large power of A, say A^t. The best way to do this is using diagonalization:

$$A^t = (SDS^{-1})^t = SDS^{-1}SDS^{-1} \dots SDS^{-1} = SD^tS^{-1}.$$

Since D is diagonal, its matrix powers are easy to compute:

$$D^t = \operatorname{diag}(\lambda_1^t, \dots, \lambda_n^t).$$

Example 17.6. Find A^t, where $A == \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$. Then

$$\begin{aligned} A^t &= SD^tS^{-1} \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .7^t & 0 \\ 0 & 1^t \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -1(.7)^t & 2 \\ 1(.7)^t & 1 \end{bmatrix} \frac{-1}{3} \begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} (.7)^t + 2 & -2(.7)^t + 2 \\ -(.7)^t + 1 & 2(.7)^t + 1 \end{bmatrix}. \end{aligned}$$

As t becomes very large this matrix approaches

$$A^\infty := \frac{1}{3} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

17.2. **Application: The Fibonacci sequence.** The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \dots$ was introduced in 1202. Each number is the sum of the two previous numbers. We will find a closed formula for the n-th Fibonacci number, using eigenvalues and eigenvectors.

To express this as a linear system, we do the following trick which is very important not only for linear algebra but also differential equations.

$$f(n+1) = f(n) + f(n-1)$$
$$f(n) = f(n)$$

What's the point of writing the second equation, which is obvious? The point is that the vector $\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$ can now be written as a matrix times $\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}$:

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

This means that

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. To find A^n, we diagonalize A.

To find the eigenvalues, we set

$$0 = \det(A - \lambda I) = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = (\lambda - (1+\sqrt{5})/2)(\lambda - (1-\sqrt{5})/2).$$

The eigenvectors are

$$\mathbf{v}_{\pm} = \begin{bmatrix} (1 \pm \sqrt{5})/2 \\ 1 \end{bmatrix}.$$

etc.

17.3. **Similarity.** Two matrices A, B are said to be *similar* if there exists an invertible matrix S such that $A = SBS^{-1}$.

Proposition 17.7. *(1) A is similar to itself. (2) If A is similar to B, then B is similar to A. (3) If A is similar to B and B is similar to C then A is similar to C.*

Theorem 17.8. *If A and B are similar, then they have the same characteristic polyonmial. As a result, they have the same eigenvalues, with the same algebraic multiplicities.*

A matrix is diagonalizable if and only if it is similar to a diagonal matrix. Let's apply this to prove the following statement:

Proposition 17.9. *The geometric multiplicity of an eigenvalue λ_i is between 1 and the algebraic multiplicity of λ_i.*

Proof. First

$$\text{geommult}\, \lambda_i = \dim(\text{nullspace}(A - \lambda_i I)) \geq 1$$

since $\det(A - \lambda_i I) = 0$.

Second, let $v_1, \ldots, v_r$ be a basis for E_{λ_i} and extend it to a basis $v_1, \ldots, v_n$ for R^n. Let S bd the matrix whose columns are $v_1, \ldots, v_n$. Note that

$$SAS^{-1} e_j = SAv_j = S\lambda_i v_j = \lambda_i e_j$$

for $j \leq r$, so SAS^{-1} is of the form

$$SAS^{-1} = \begin{bmatrix} \lambda_i I_r & * \\ 0 & * \end{bmatrix}$$

that is, block upper-triangular with $\lambda_i I_r$ in the upper-left corner. Hence $\det(A - \lambda I) = \det(SAS^{-1} - \lambda I)$ has at least r copies of $\lambda - \lambda_i$. So the algebraic multiplicity is at least r. $\square$

18. Complex eigenvalues

To define complex numbers, we suppose that -1 has a square root, called i, the *imaginary unit*, so

$$i^2 = -1.$$

This is similar to how negative numbers are introduced: $-x$ is the number which satisfies the equation $-x + x = 0$.

An imaginary number is any real multiple bi of the imaginary unit i. A complex number is the sum of a real number plus an imaginary number. The sum of complex numbers is defined by summing the real and imaginary parts

$$(5 + 2i) + (3 - 4i) = 8 - 2i.$$

Differences are similar:

$$(5 + 2i) - (3 - 4i) = 2 + 6i.$$

The product of complex numbers is again a complex number, using that $i^2 = -1$:

$$(5 + 2i)(3 - 4i) = 15 - 20i + 6i - 8i^2 = 15 - 14i - 8(-1) = 23 - 14i.$$

Geometrically, complex numbers are represented as points in the complex plane, which has horizontal axis the *real axis* and vertical axis the *imaginary axis*.

Sum and subtraction of complex numbers is the same as addition and subtraction of two-vectors.

The *complex conjugate* of a complex number $z = a + bi$ is the reflection of that number over the real axis,

$$\overline{z} = a - bi.$$

The *norm* $|z|$ of a complex number $z = a + bi$ is the length of the corresponding 2-vector,

$$|z| = \sqrt{a^2 + b^2}.$$

The norm can also be defined using the conjugate:

$$z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 \quad \text{so} \quad |z| = \sqrt{z\overline{z}}.$$

This gives us a way to define inverses of complex numbers: We have

$$z\frac{\overline{z}}{z\overline{z}} = 1 \quad \text{so} \quad z^{-1} = \frac{\overline{z}}{z\overline{z}}.$$

Example 18.1. Find the inverse of $5 + 2i$.

$$(5+2i)^{-1} = \frac{5-2i}{(5+2i)(5-2i)} = \frac{5-2i}{25+4} = \frac{5}{29} - \frac{2}{29}i.$$

18.1. **Polar form.** The geometric meaning of multiplication is best explained by the *polar form* of a complex number. Define the *argument* $\arg z$ to be the angle θ between z and the positive real axis. Define $r = |z|$ the *modulus* of z.

Then the adjacent (resp.) side in the picture is

$$a = r\cos(\theta), b = r\sin(\theta).$$

The polar form of z is

$$z = r\cos(\theta) + r\sin(\theta)i.$$

The old way $z = a + bi$ is called *Cartesian form.*

Example 18.2. Find the polar form of $1, i, 1+i, -1, -1-i$.

From the picture: For $z = 1$ we have $r = 1, \theta = 1$. For $z = i$ we have $r = 1, \theta = \pi/2$. For $z = 1+i$ we have $r = \sqrt{2}, \theta = \pi/4$. etc.

Proposition 18.3. *Complex conjugation in polar form reverses the sign of the angle θ: If $z = r\cos(\theta) + r\sin(\theta)i$ then $\overline{z} = r\cos(-\theta) + r\sin(-\theta)i$.*

Proof. $\overline{z} = r\cos(\theta) - r\sin(\theta)i = r\cos(-\theta) + r\sin(-\theta)i$. □

Multiplication in polar form is simpler than in Cartesian form. Recall the Taylor series expansions

$$e^x = 1 + x + x^2/2! + x^3/3! + \dots.$$

$$\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! \dots$$

$$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! \dots$$

From the Taylor series for e^x we get

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + (i\theta)^2/2! + (i\theta)^3/3! + \dots \\ &= (1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \dots) + (\theta - \theta^3/3! + \theta^5/5! + \dots)i \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

This shows

Theorem 18.4 (Euler)**.** $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Now we can re-write the polar form

$$z = r\cos(\theta) + r\sin(\theta)i = re^{i\theta}.$$

This implies the following geometric interpretation of multiplcation of complex numbers.

Proposition 18.5. *Multiplication of complex numbers is given by multiplying the lengths and adding the angles. That is, if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then*

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}.$$

18.2. **Application: Angle-sum formulas.** It's easy to derive from Euler's formula the formulas for the cosine or sine of the sum of two, three, or more angles. For instance,

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

On the other hand,

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= e^{i\theta_1}e^{i\theta_2} \\ &= \cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \end{aligned}$$

Matching up the real and imaginary parts, we get

Proposition 18.6. $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2))$ *and* $\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)$.

18.3. **Application: Powers of complex numbers.** The best way to find a large power of a complex number is to first write it in polar form.

Example 18.7. Find z^{20} where $z = 1 + i$. Since the length $r = \sqrt{2}$ and the angle $\theta = \pi/4$, the polar form is $z = \sqrt{2}e^{i(\pi/4)}$. So

$$z^{20} = \sqrt{2}^{20}(e^{i(\pi/4)})^{20} = 2^{10}e^{i5\pi} = 1024(\cos(5\pi) + i\sin(5\pi)) = 1024(-1) = -1024.$$

18.4. **The fundamental theorem of algebra.** If we allow complex numbers, then any quadratic polynomial now has exactly two roots (counted with multiplicity) since $\sqrt{b^2 - 4ac}$ always makes sense as a complex number.

Example 18.8. Find the roots of $f(x) = x^2 + x + 3$. Ans $x = (-1 \pm \sqrt{-8})/2$.

In fact, we have the following theorem.

Theorem 18.9 (Fundamental Theorem of Algebra)**.** *If $f(x) = c_0 + c_1 x + \ldots + c_n x^n$ is a polynomial of degree n, then f can be factored $f(x) = c_n(x - z_1)(x - z_2) \ldots (x - z_n)$ where $z_1, \ldots, z_n$ are complex numbers. That is, any degree n polynomial has exactly n roots, counted with multiplicity.*

Corollary 18.10. *Any $n \times n$ matrix has exactly n eigenvalues, counted with algebraic multiplicity, if we allow complex eigenvalues.*

Proof. Apply the fundamental theorem of algebra to the characteristic polynomial $\det(A - \lambda I)$. □

If we have a complex eigenvalues, we can define complex eigenvectors just as before.

Example 18.11. Find the complex eigevalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Ans. The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 + 1$ which has roots $\lambda_\pm = \pm i$. The eigenspaces are: $\lambda = i$.

$$E_i = \text{nullspace}(A - iI) = \text{nullspace} \begin{bmatrix} -i1 & \\ 0 & -1ii \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Similarly, the eigenspace E_{-i} is the span of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

The geometric multiplicity of a complex eigenvector λ is the dimension of the (complex) subspace E_λ. It is at least 1 and at most the algebraic multiplicity.

Corollary 18.12. *A matrix is diagonalizable over the complex numbers if and only if the geometric multiplicity equals the algebraic multiplicity of λ, for each complex eigenvalue λ.*

Example 18.13. Find the eigenvalues and eigenvectors for the "shift matrix"

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

For complex vectors v, the norm $\|v\|$ is defined by

$$\|v\| = v \cdot \overline{v}.$$

If $v = [a_1 + b_1 i \ a_2 + b_2 i \ \ldots \ a_n + b_n i]$ then

$$\|v\| = \sqrt{a_1^2 + b_1^2 + \ldots a_n^2 + b_n^2}.$$

The only vector with norm 0 is the zero vector.

19. Symmetric matrices and quadratic forms

One class of matrices for which complex eigenvectors and eigenvalues never appear is symmetric matrices.

Proposition 19.1. *If A is symmetric and real, $A = A^T = \overline{A}$, then all the eigenvalues are real.*

Proof. If v is a (possibly complex) eigenvector with (possibly complex) eigenvalue λ then either $\lambda = 0$, which is real, or

$$\begin{aligned} \|v\| &= v^T\overline{v} &&= (\frac{Av}{\lambda})^T\overline{v} \\ &= \tfrac{1}{\lambda}v^TA^T\overline{v} &&= \frac{1}{\lambda}v^T\overline{A}\overline{v} \\ &= \tfrac{1}{\lambda}v^T\overline{Av} &&= \frac{1}{\lambda}v^T\overline{\lambda v} \\ &= \tfrac{1}{\lambda}v^T\overline{\lambda}\overline{v} &&= \frac{\overline{\lambda}}{\lambda}v^T\overline{v} \\ &= \tfrac{\overline{\lambda}}{\lambda}\|v\|^2 \end{aligned}$$

Since $\|v\| \neq 0$, we must have $\lambda = \overline{\lambda}$, which means that λ is on the real axis. □

Proposition 19.2. *If A is a symmetric real matrix, and Av and w are perpendicular vectors, then so are v and Aw.*

Proof. $0 = (Av) \cdot w = (Av)^T w = v^T A^T w = v^T Aw = v \cdot (Aw)$. □

Theorem 19.3. *If A is a symmetric $n \times n$ matrix then A is diagonalizable, and there exists an orthonormal basis of eigenvectors $v_1, \ldots, v_n$.*

Proof. By induction on the size n. Let v_1 be an eigenvector normalized to have length one, and $v_2, \ldots, v_n$ vectors so that $v_1, \ldots, v_n$ is an orthonormal basis. Then $\lambda_1 v_1 = Av_1$ is perpendicular to $v_j, j > 1$, which implies that Av_j is perpendicular to v_1. Hence Av_j is a combination of the vectors $v_2, \ldots, v_n$.

Let S be the matrix with columns $v_1, \ldots, v_n$. Since these form an orthonormal basis, S is orthogonal, $S^{-1} = S^T$. Then

$$S^T ASe_1 = S^T Av_1 = \lambda_1 e_1$$

and

$$S^T ASe_j = S^T Av_j$$

is a combination of the vectors $e_2, \ldots, e_n$. Therefore, $S^T AS$ has block diagonal form $\mathrm{diag}(\lambda_1, A_1)$ for some $n-1 \times n-1$-matrix A_1. Since A is symmetric and S is orthogonal, $S^T AS$ is symmetric. So A_1 is symmetric as well. By the inductive hypothesis, R^{n-1} has an orthonormal basis of eigenvectors $w_2, \ldots, w_n$ for A_1. So $e_1, w_2, \ldots, w_n$ is an orthonormal eigenbasis for $S^T AS$. But then the vectors $S^T e_1, \ldots, S^T w_n$ are an orthonormal eigenbasis for A. □

Example 19.4. $ATT, MCI, Spring$ compete for customers. Each offers pretty crummy service, and so loses 20 per cent of its customers to each of its competitors, each month. If $a(t), m(t), s(t)$ denote the number of customers in month t, then

$$\begin{bmatrix} a(t+1) \\ m(t+1) \\ s(t+1) \end{bmatrix} = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}.$$

Because the columns of this matrix sum up to one, we know that $\lambda = 1$ is an eigenvalue. To find the others, we long divide $(\lambda - 1)$ into the characteristic polynomial ....

19.1. **Quadratic Forms.** Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. A *quadratic form* if a function $q(x_1, \ldots, x_n)$ that is degree two in the variables $x_1, \ldots, x_n$.

Example 19.5.

$$q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$$

is a quadratic form in two variables x_1 and x_2. We can try to graph the function q in three dimensions. First we can graph the function when $x_1 = 0$, and then when $x_2 = 0$. You might think that the function is always "going up". But it's not. It actually looks like a saddle.

To prove this, let's write q in matrix form. Let Q be the matrix whose diagonal entries are the coefficients of x_1^2 and x_2^2, and whose off-diagonal entries are 1/2 the coefficient of x_1, x_2:

$$Q = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Let's find the eigenvectors and eigenvalues for Q. The characteristic polynomial is

$$\det(Q - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

so the eigenvalues are

$$\lambda = -1, 3.$$

We can find the eigenvalues: For $\lambda = 1$ the eigenspace is

$$E_{-1} = \text{nullspace} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{5}.$$

$$E_3 = \text{nullspace} \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{5}.$$

So Q can be diagonalized

$$Q = SDS^T, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

This means that if we define new coordinates

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = S^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then

$$q(y_1, y_2) = \mathbf{y}^T D \mathbf{y} = -y_1^2 + 3y_2^2.$$

This quadratic form is easy to understand; it is an upside down saddle.

Another way of graphing the quadratic form is to draw it's level sets

$$q(y_1, y_2) = c.$$

For the example above, these are hyperbolas.

Example 19.6. Graph the quadratic form $q(x_1, x_2) = 2x^2 + 2xy + 2y^2$. Describe the level set $q(x_1, x_2) = 4$.

20. Discrete Time Dynamical Systems and Recursive Sequences

A dynamical system is a mathematical model for the time evolution of "real life" quantities, for instance, in physics, chemistry, biology, economics, etc. A *discrete time* dynamical system means that we suppose that time takes only discrete values, say days, months, years, etc. Another type of dynamical system is *continuous time*; we won't discuss those here.

20.1. **An economic example.**

Example 20.1. A typical example is the Coke/Pepsi example discussed earlier. Recall: Suppose $c(t)$ (resp. $p(t)$) is the number of Coke (resp. Pepsi) drinkers at time t months. Suppose each month, 10 percent of the Coke drinkers switch to become Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as t goes to infinity?

t	p	c
0	0	100
1	20	80
2	34	66

To set this up as a linear algebra problem we write

$$c(t+1) = .9c(t) + .2p(t)$$
$$p(t+1) = .1c(t) + .8p(t)$$

or in matrix form

$$\mathbf{x}(t+1) = A\mathbf{x}(t) \text{ where } A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

This implies that

$$\mathbf{x}(t) = A\mathbf{x}(t-1) = A^2\mathbf{x}(t-2) = \ldots = A^t\mathbf{x}(0)$$

The *state* of the system in any month t is described by the vector of Coke/Pepsi drinkers

$$\mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

The *time evolution* of the system is an equation for $\mathbf{x}(t+1)$ in terms of $\mathbf{x}(t)$:

$$\mathbf{x}(t+1) = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \mathbf{x}(t).$$

The matrix in this equation is the *time evolution matrix*.

Example 20.2. Two companies are competing for customers. Each year, company A loses 60 percent of its customers to company B, while company B each year loses 70 percent of its customers to company A. (Clearly neither of the company's produce a very high quality product!) (a) Write down the state vector and time evolution matrix for this system. That is, represent the system in the form $x(t+1) = Ax(t)$, for some matrix A. (b) Find the diagonalization of the matrix A. (c) Suppose that initially, 100 customers are with company A and none with company B. Find a formula for the number of customers with company A at time t. (d) How many customers does A have, for t very large. (e) Show the evolution of the system on the graph with axes A,B.

20.2. **Biology examples.**

Example 20.3. The model for the flu epidemic we discussed earlier. Suppose that in a population of 80 students, at any point in time there are w well students, s sick students, and i students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off.

The state vector is the number of well, sick, and immune students

$$\mathbf{x}(t) = \begin{bmatrix} w(t) \\ s(t) \\ i(t) \end{bmatrix}$$

and the time evolution matrix is

$$A = \begin{bmatrix} .8 & 0 & 1 \\ .2 & .5 & 0 \\ 0 & +.5 & 0 \end{bmatrix}.$$

To solve for $\mathbf{x}(t)$, we find the eigenvectors and eigenvalues .....

Example 20.4. Let's consider a new example, a population model. Suppose that we consider a population of baby, adult, and retired rabbits. The baby rabbits (resp. adult rabbits) mature into adult (resp. retired) rabbits after one year; life span is three years. Each adult rabbit has 3/4 baby rabbit, on average, each year. Each retired rabbit, has on average, 1/4 a baby each year. Find the state vector and the time evolution matrix for this situation. Find the ratio of baby, adult, and retired rabbits in the limit $t \to \infty$.

20.3. **Recursive sequences.** A *recursive sequence* is a sequence of numbers where the n-th number is defined by a formula involving previous numbers. The most famous example, the Fibonacci sequence

$$f(n+1) = f(n) + f(n-1), \quad f(n) = 1, 1, 2, 3, 5, 8, 13, \ldots$$

was introduced in 1202 in the context of a population model similar to the one above. Each number is the sum of the two previous numbers. We will find a closed formula for the n-th Fibonacci number, using eigenvalues and eigenvectors.

To express this as a linear system, we do the following trick which is very important not only for linear algebra but also differential equations.

$$f(n+1) = f(n) + f(n-1)$$

$$f(n) = f(n)$$

What's the point of writing the second equation, which is obvious? The point is that the vector $\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix}$ can now be written as a matrix times $\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}$:

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

This means that

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. To find A^n, we diagonalize A.

To find the eigenvalues, we set

$$0 = \det(A - \lambda I) = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = (\lambda - (1+\sqrt{5})/2)(\lambda - (1-\sqrt{5})/2).$$

The eigenvectors are

$$\mathbf{v}_\pm = \begin{bmatrix} (1 \pm \sqrt{5})/2 \\ 1 \end{bmatrix}.$$

etc.

Example 20.5. Let's look now at some other recursive formula, for instance

$$f(n+1) = f(n) - f(n-1)$$

which gives the sequence

$$1, 1, 0, -1, -1, 0, 1, 1, \dots$$

Find a closed formula for $f(n)$.

To solve the equation we introduce a second equation

$$f(n) = f(n)$$

so that we get a system of linear equations

$$\begin{bmatrix} f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = (1-\lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1$$

which has roots

$$\lambda_\pm = \frac{1 \pm \sqrt{-3}}{2} = e^{\pm 2\pi i/3}.$$

This means that the matrix A is diagonalizable

$$A = SDS^{-1}, \quad D = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3}. \end{bmatrix}$$

If n is a multiple of 3 then

$$A^n = SD^nS^{-1} = SS^{-1} = I.$$

This explains why the sequence is periodic with period 3.

Example 20.6. Define a sequence $f(n)$ by $f(n+1) = 2f(n) + f(n-1)$. Find a closed formula for $f(n)$ using linear algebra.

21. Factorization philosophy

Let's look again at three of the most important algorithms in the course. (1) Elimination, (2) Gram-Schmidt orthogonalization, (3) diagonalization.

Each of these can be though of as procedures for *factoring* the original matrix A. A *factorization* of A is an expression for A as a product of matrices, each of which has a special property. For instance, diagonalization is equivalent to factoring the matrix

$$A = SDS^{-1}$$

into the product of an invertible matrix S, a diagonal matrix D, and S^{-1}.

21.1. **Elimination is LU factorization.**

When we find the reduced row-echelon form of a matrix A, we reduce A to an upper triangular matrix, it's reduced row-echelon form. Since this matrix is upper triangular, let's call it U.

Example 21.1. We find the rref of $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}$. First, we multiply the first row by $1/2$. Then we subtract three times the first row from the second to get

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To get the row of zeros at the bottom, we switch rows 2 and 3:

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, we multiply row 2 by $1/2$ to get a ref.

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each row operation can be represented by multiplying by an elementary matrix on the left. There are three types of *elementary matrices*:

(1) $E_{ij}(c)$ is equal to the identity matrix, except for the ij-th entry which is a constant c. Multiplying on the left by E_{ij} has the effect of subtracting c times row j from row i.

Example 21.2. $E_{21}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix. We gave $E_{21}(-2) \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ -2r_1 + r_2 \\ r_3 \end{bmatrix}$.

(2) $E_{ii}(c)$ is equal to the identity matrix, except for the ii-th entry which is a number c. Multiplying on the left by $E_{ii}(c)$ has the effect of multiplying row i by c.

Example 21.3. $E_{22}(1/2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $E_{22}(1/2)\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ (1/2)r_2 \\ r_3 \end{bmatrix}$.

(3) P_{ij} is the identity matrix, except with the 1's in rows i, j switched. It is also called the *transposition matrix* for i,j. Multiplying by P_{ij} on the left has the effect of switching rows i and j.

Example 21.4. $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. $P_{13}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_3 \\ r_2 \\ r_1 \end{bmatrix}$.

Example 21.5. We represent elimination for $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}$ using elimination. First, we multiply by E_{11} with $c = 1/2$ to get

$$E_{11}(1/2)A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix}.$$

We multiply by E_{21} with $c = -3$ to subtract three times the first row from the second:

$$E_{21}(-3)E_{11}(1/2)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 11 \end{bmatrix}.$$

We multiply by E_{31}(-3) with $c = -3$ to subtract three times the first row from the third:

$$E_{31}(-3)E_{21}(-3)E_{11}(1/2)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We multiply by P_{23} to switch rows 2 and 3:

$$P_{23}E_{31}E_{21}E_{11}(1/2)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, we multiply by $E_{22}(1/2)$ with $c = 1/2$ get a ref.

$$E_{22}(1/2)P_{23}E_{31}(-3)E_{21}(-3)E_{11}(1/2)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{ref}(A).$$

If we want to, we can put all the E and P matrices on the right, since they are invertible. Here are the inverses of the elementary matrices:

(1) the inverse of the matrix $E_{ij}(c)$ is $E_{ij}(-c)$ (since the opposite of adding c times row j is subtracting)

(2) the inverse of the matrix $E_{ii}(c)$ is $E_{ii}(1/c)$ (since multiplying by c is reversed by multiplying by $1/c$)

(3) the inverse of P_{ij} is P_{ij} (since switching twice gets back the original.)

So

$$A = E_{11}(2)E_{21}(3)E_{31}(3)P_{23}E_{22}(2)\,\mathrm{ref}(A).$$

The matrix $\mathrm{ref}(A)$ is denoted U, since it is upper triangular. The matrix $E_{11}(2)E_{21}(3)E_{31}(3)P_{23}E_{22}(2)$ can be written

$$E_{11}(2)E_{21}(3)E_{31}(3)P_{23}E_{22}(2) = E_{11}(2)E_{21}(3)E_{31}(3)E_{33}(2)P_{23}$$

(since P_{23} switches rows 2 and 3, switching the order changes $E_{22}(2)$ to $E_{33}(2)$.)

The product of the E's is lower triangular, it is denoted L. The product of the P_{ij}'s (in this case, there is just one) is denoted P; it is always a permutation matrix.

It is not hard to see that the same procedure works for any matrix, of any size. When doing eliminiation one can always do the switches first (or last); then the P_{ij}' will be at the beginning (resp. in the middle). This shows:

Theorem 21.6. *Any matrix A admits a factorization $A = LPU$, where L is lower triangular, P is a permutation matrix, and U is upper triangular (and is a* ref *for A.)*

Let's use this version of elimination to give another proof of how the determinant changes under row-reduction.

(1) The determinant changes sign when rows are switched.

Proof: $\det(P_{ij}A) = \det(P_{ij})\det(A) = -\det(A)$, since $\det(P_{ij}) = -1$.

(2) The determinant is unchanged when a multiple of one row is added to another.

Proof: $\det(E_{ij}A) = \det(E_{ij})\det(A) = \det(A)$, since $\det(E_{ij})$ is the product of the diagonal entries which equals 1.

(3) The determinant is multiplied by c, if row i is multiplied by c.

Proof $\det(E_{ii}A) = \det(E_{ii})\det(A) = c\det(A)$.

Two final remarks:

(1) The LPU factorization is not in general unique. For instance, one can use elimination to get to a number of different matrices U. But, if one requires that the U is a row-echelon form of A, then the factorization *is* unique.

(2) If A is symmetric and positive (has all positive eigenvalues), then one can find a factorization $A = LU$ so that $L = U^T$.

21.2. **Gram-Schmidt is QR factorization.** We can apply the same sorts of ideas to Gram-Schmidt Factorization. For instance, let's apply Gram-Schmidt to the three vectors

$$v_1 = [1\ 1\ 0]\,, v_2 = [1\ 0\ 1]\,,\ v_3 = [0\ 1\ 1]\,.$$

Then

$$u_1 = [1\ 1\ 0]\,/\sqrt{2},$$

$$v_2 \mapsto v_2 - (v_2 \cdot u_1)u_1 = [1\ 0\ 1] - \frac{1}{2}[1\ 1\ 0] = [1/2\ \ -1/2\ 1] \mapsto [1\ \ -1\ 2]$$

$$u_2 = [1\ \ -1\ 2]\,/\sqrt{6}$$

Note u_2 is obtained from $[1/2\ -1/2\ 1]$ by dividing by the length, $\sqrt{3/2}$.

$$v_3 \mapsto v_3 = (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 = [0\ 1\ 1] - \frac{1}{2}[1\ 1\ 0] - \frac{1}{6}[1\ -1\ 2] = [-2/3\ 2/3\ 2/3] \mapsto [-1\ 1\ 1].$$

$$u_3 = [-1\ 1\ 1]/\sqrt{3}.$$

Each of these operations can be thought of as multiplication of the matrix

$$A = [v_1\ v_2\ v_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

by an elementary matrix on the *right.*

First, we multiply A by $E_{11}(1/\sqrt{2})$ to divide the first column by $\sqrt{2}$.

$$AE_{11}(1/\sqrt{2}) = \begin{bmatrix} 1/\sqrt{2} & 1 & 0 \\ 1/\sqrt{2} & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

If we multiply on the right by $E_{12}(-1/\sqrt{2})$, this subtracts $1/\sqrt{2}$ times column one from column 2:

$$AE_{11}(1/\sqrt{2})E_{12}(-1/\sqrt{2}) = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 0 \\ 1/\sqrt{2} & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

If we multiply on the right by $E_{22}(1/\sqrt{3/2})$, we get

$$AE_{11}(1/\sqrt{2})E_{12}(-1/\sqrt{2})E_{22}(1/\sqrt{3/2}) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 1 \\ 0 & 2/\sqrt{6} & 1 \end{bmatrix}.$$

Then, we multiply on the right by $E_{13}(-1/\sqrt{2})$ and $E_{13}(-1/\sqrt{6})$ to get

$$AE_{11}(1/\sqrt{2})E_{12}(-1/\sqrt{2})E_{22}(1/\sqrt{3/2})E_{13}(-1/\sqrt{2})E_{13}(-1/\sqrt{6}) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/3 \\ 0 & 2/\sqrt{6} & 2/3 \end{bmatrix}.$$

Finally, multiply by $E_{33}(\sqrt{3}/2)$ to get

$$AE_{11}(1/\sqrt{2})E_{12}(-1/\sqrt{2})E_{22}(1/\sqrt{3/2})E_{13}(-1/\sqrt{2})E_{23}(-1/\sqrt{6})E_{33}(\sqrt{3}/2) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

This final matrix is called Q; it's columns form an orthonormal set of vectors. We can move the E's on the right, (taking inverses switches order)

$$A = QE_{33}(2/\sqrt{3})E_{23}(1/\sqrt{6})E_{13}(1/\sqrt{2})E_{22}(\sqrt{3/2})E_{12}(1/\sqrt{2})E_{11}(\sqrt{2}).$$

The matrix that is the product of the E's is called R:

$$R = E_{33}(2/\sqrt{3})E_{23}(1/\sqrt{6})E_{13}(1/\sqrt{2})E_{22}(\sqrt{3/2})E_{12}(1/\sqrt{2})E_{11}(\sqrt{2}).$$

Since the E's are upper triangular, so is R.

The same procedure works for any vectors $v_1, \ldots, v_m$. This shows

Theorem 21.7. *Any matrix A can be factored $A = QR$, where the columns of Q form an orthonormal set and R is upper triangular.*

Note that Q will not be an orthogonal matrix unless A is square.