ELEMENTARY GEOMETRY SUMMARY

CHRIS WOODWARD, FALL 2009

Contents

This is not quite finished but I am passing it out now so you have plenty of time to study.

1. Lines, Rays, Angles

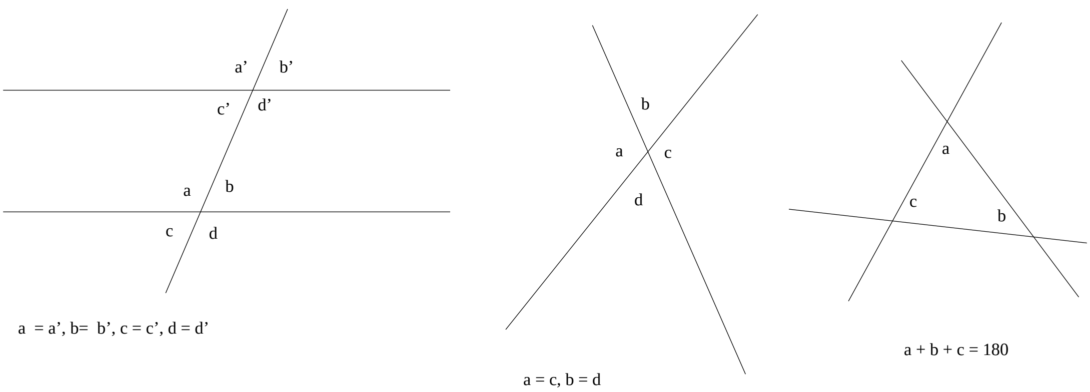
An *angle* is a pair of rays starting at the same point. Two angles are *congruent* if they are related by a rigid motion. Angles are often measured in *degrees*, where 360 represents a full turn. An angle is *acute* if it less than 90 degrees, *right* if it is 90 degrees, and *obtuse* if it is more than 90 degrees.

Three facts about angles:

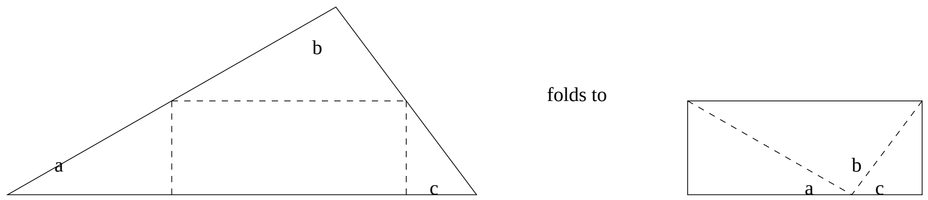
(1) If two parallel lines are bisected by a third line, then the angles at the first intersection point are equal to those at the second intersection point. (Parallel postulate).

(2) Opposite angles are equal.

(3) If three lines all bisect each other, then the interior angles at the intersection points sum to 180 degrees.

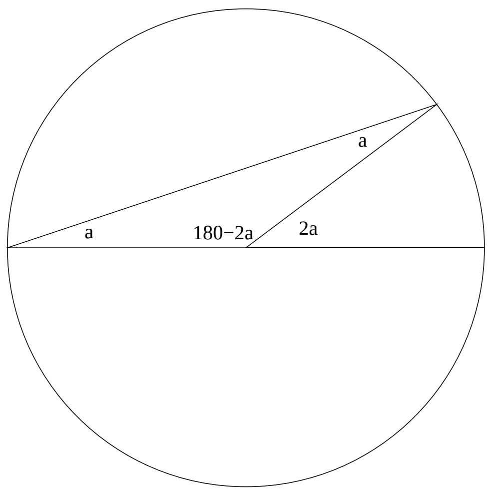


Here is a visual proof that the angles in a triangle sum to 180:



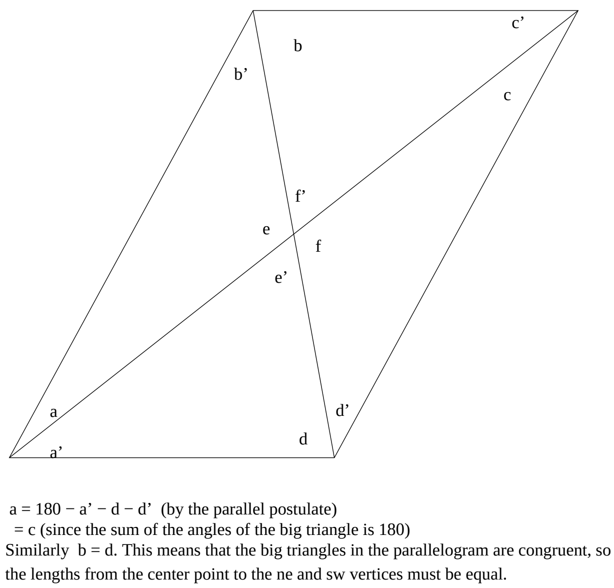
These three facts can be used to prove a wide variety of facts in geometry.

Example: The central angle is twice the inscribed angle of a triangle on a circle.

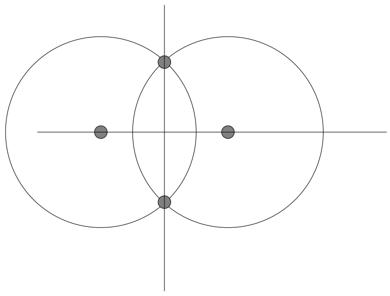


The figure shows the case that one side of the triangle passes through the center of the circle; the general case is a combination of two of these cases.

Example: The diagonals of a parallelogram bisect. (The congruence in the argument below uses the angle-side-angle criterion.)

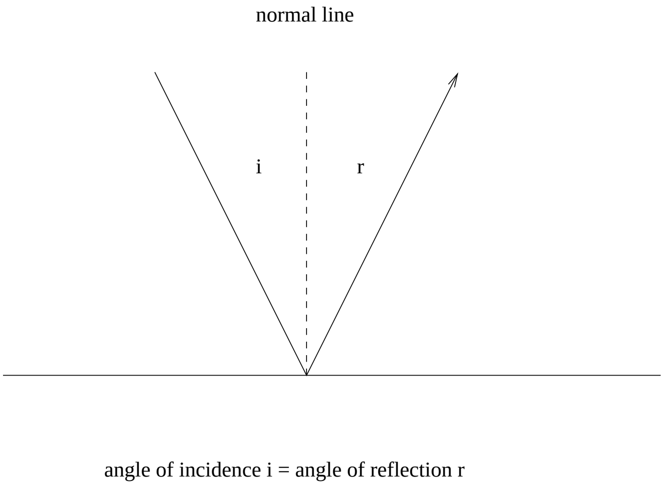


Classically parallel and perpendicular lines were constructed using *ruler-straightedge constructions*. For example, to construct a right angle too a line, use a compass around two points on the line to draw two circles. The two intersection points of the circles are contained in a perpendicular line.

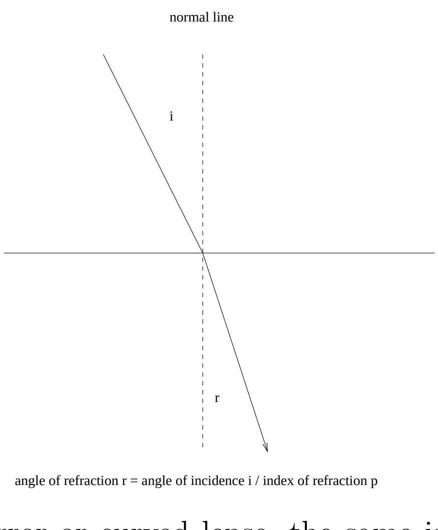


Angles in optics:

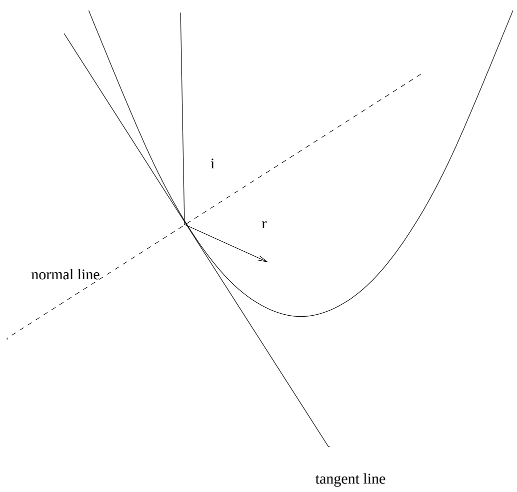
When light hits a smooth mirror, it reflects with *angle of reflection* equal to *angle of incidence* (both measured from the *normal*, that is, the line perpendicular to the tangent line.

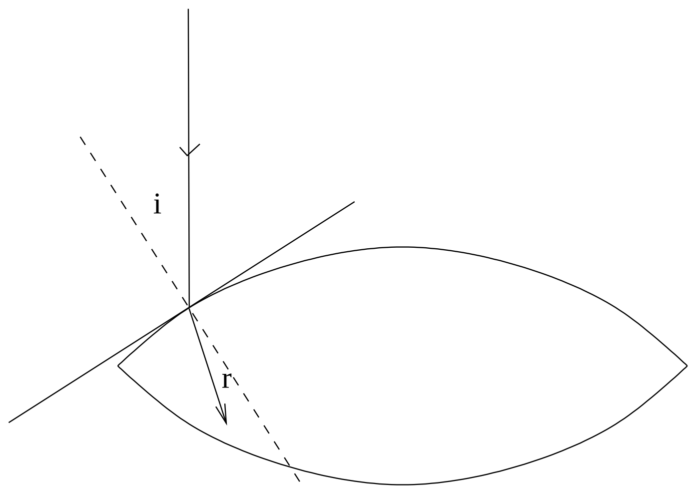


When light hits another transparent object, such as water or glass, the part that passes through passes with *angle of refraction* equal to the *angle of incidence* divided by the *index of refraction*, about 1.5 for glass.



For a curved mirror or curved lense, the same is true but you first have to find the *tangent line*:



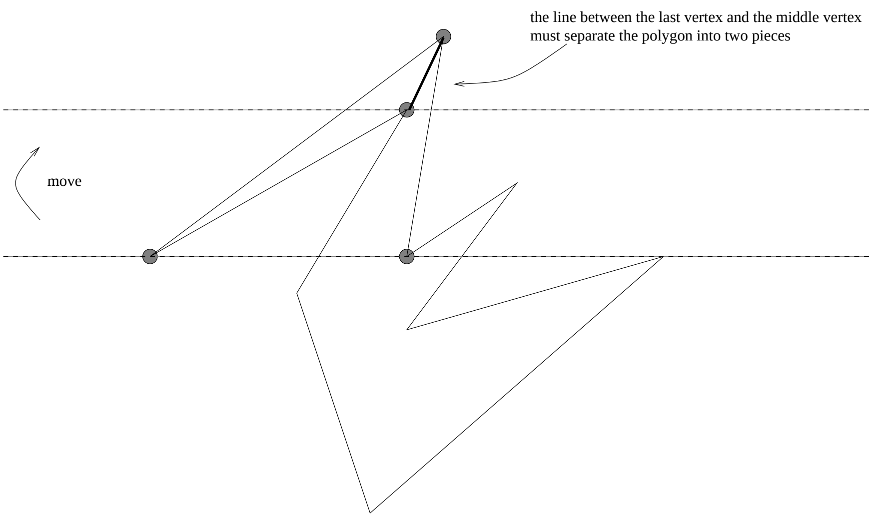


It's not hard to see that the image on the other side of such a lense will appear larger than it would otherwise. So the lense *magnifies* the image as it passes through the lense.

2. Polygons and two dimensional shapes

A *polygon* is a closed shape in the plane formed by a sequence of line segments, such that any two consecutive line segments intersect in an endpoint.

Any n-gon can be divided into $n-2$ triangles. To find a triangle inside an n-gon, pick any three consecutive vertices and draw the line between the first and third. Either the line splits the polygon into two pieces, one of which is a triangle, or it goes outside the polygon. If it goes outside, then move it until it passes through the second to last vertex in that direction. Then the line between the second to last vertex and the last vertex must be completely containd in the polygon,

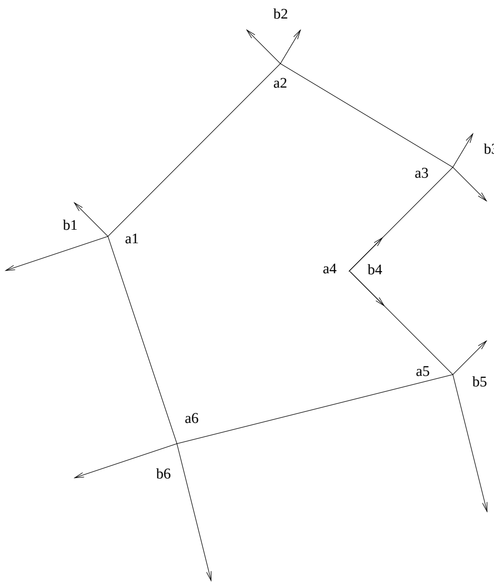


Proof that any polygon can be dividd into triangles: The argument above shows that the polygon can be divided into two pieces. Now keep going until every piece is a triangle.

The sum of angles in an n-gon is $180(n-2)$.

One proof: divide into triangles and use the sum of the angles in any triangle is 180.

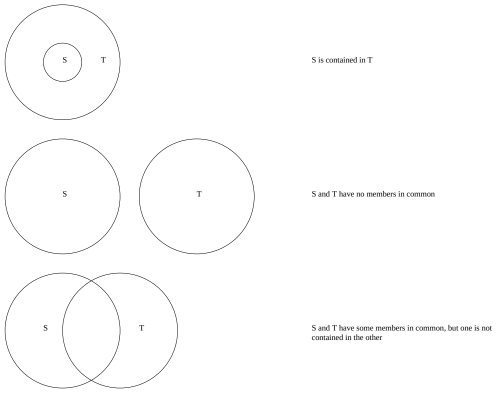
Another proof: the angles between the *outward normals* of any polygon sum to 360. Each outward angle b_i is related to the inner angle by $180 - a_i$. So $\sum a_i = \sum(180 - b_i) = n180 - \sum b_i = n180 - 360 = (n-2)180$.



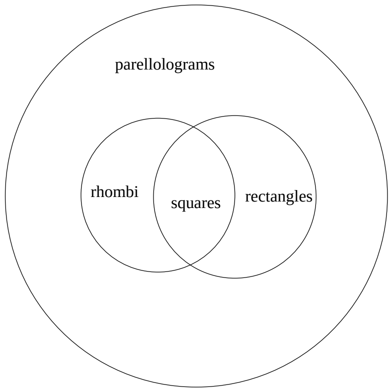
A *triangle* is a 3-gon. A triangle is *isosceles* if it hsa two equal sides, *equilateral* if all sides are equal, and *scalene* otherwise.

A *quadrilateral* is a 4-gon. A *square* has all sides equal and all angles right; *parallelogram* has two sets of parallel sides; *trapezoid* has one set of parallel sides; *rhombus* has all sides equal; *rectangle* has all right angles.

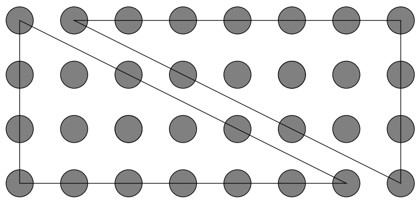
A *Venn diagram* shows the relationship between different groups: if one group is contained in another, there is a bigger circle arond the smaller. If two groups have no members in common, there are two circles not intersecting. If the two groups intersect, the two circles intersect.



So, for example, squares, rectangles, rhombi, and parallelograms are related as follows.



Polygons often arise in counting problems. For example, to find a formula for $1 + 2 + \ldots + n$ geometrically, we realize that it is the number of points in a right triangle with side lengths $n, n, 2n$. Putting two such triangles together gives a rectangle of size $n \times (n + 1)$, so the total number of points in the rectangle is $n(n + 1)$, so the number of points in the triangle is $1 + 2 + \ldots + n = n(n + 1)/2$. Similarly $1 + 3 + 5 + 7 + \ldots + (2n + 1)$ is half the number of points in a $n \times 2n$ rectangle, hence equal to n^2.

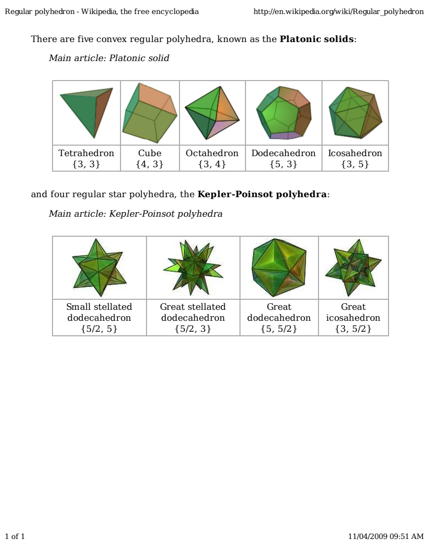


A circle (resp. sphere) of radius r around a point p is the set of points in the plane (resp. space) whose distance to p is r. The inside of the circle is the set of points whose distance is *less than* r and the outside is the set of points whose distance is *greater than* r.

3. Polyhedra and other three-dimensional shapes

A *polyhedron* is a closed shape made of polygons (the *faces* of the polyhedron), any two of which meet in a vertex or edge, if at all.

A polyhedron is *convex* if the line segment between any two points inside the polyhedron stays inside the polyhedron. A polyhedron is *regular* if the faces and vertices are all congruent.



Regular polyhedron - Wikipedia, the free encyclopedia http://en.wikipedia.org/wiki/Regular_polyhedron

There are five convex regular polyhedra, known as the **Platonic solids**:

Main article: Platonic solid

Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}

and four regular star polyhedra, the **Kepler-Poinsot polyhedra**:

Main article: Kepler-Poinsot polyhedra

Small stellated dodecahedron {5/2, 5}	Great stellated dodecahedron {5/2, 3}	Great dodecahedron {5, 5/2}	Great icosahedron {3, 5/2}

1 of 1 11/04/2009 09:51 AM

There are 5 regular convex polyhedra (tetrahedra, hexahedra, octahedra, icosahedra, and dodecahedra) and 9 regular polyhedra.

A *cylinder* on a base shape is obtained by "going up at an angle" from the base. If the angle is a right angle, the cylinder is *right*, otherwise the cylinder is *oblique*. A *prism* is a cylinder on a polyhedral base. A *cone* is obtained from a base shape by "going up to a point".

4. Similarity and congruence

In a plane suppose we have chosen an *origin* point and two *coordinate axes*: perpendicular lines passing through the point and marked with unit distances. Any point in the plane is given by a pair of real numbers, called the *coordinates*. Often the coordinate axes are labelled x, y.

A *rigid motion* is one that preserves distances. There are three kinds of rigid motions:

(i) translations (slides)
(ii) rotations (turns)
(iii) reflections (flips)

(More precisely, any rigid motion is a combination of these three; however, the combination of a translation and a reflection is not on the list above and is called a *glide-reflection*.)

The three types combine as follows: (a) translation followed by translation is a translation (b) rotation followed by a rotation is a rotation (c) rotation followed by translation is a rotation (d) reflection followed by reflection is a rotation

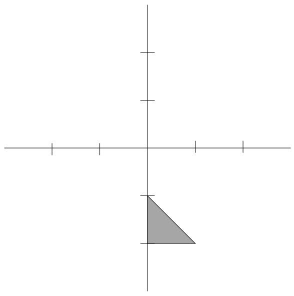
A *dilation* from a point p by a factor s is a motion of the plane that changes the distance of any point to p by the factor s, and leaves the direction the same.

Tranlation of a polyhedron can be computed by drawing parallel lines through each vertex in the direction of the translation and travelling the given distance.

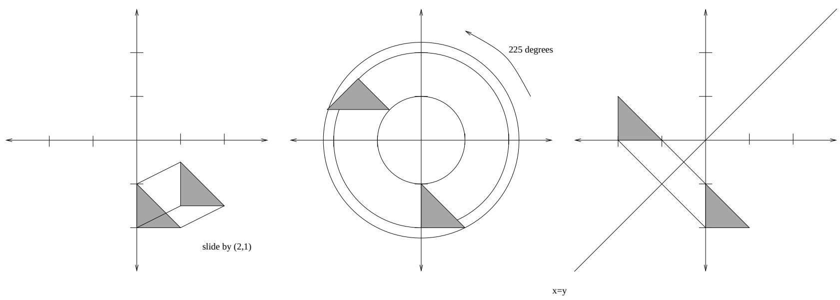
Rotation of a polyhedron can be computed by drawing circles through each point of the polyhedron and travelling the given angle.

Reflection of a polyhedron can be computed by drawing lines perpendicular to the reflection line and travelling the same distance to the other side of the reflection line.

Example: Compute the (i) translation by (2,1) (ii) rotation around 0 by 225 degrees (iii) reflection over the line $x = y$ of the figure below.



Answer:

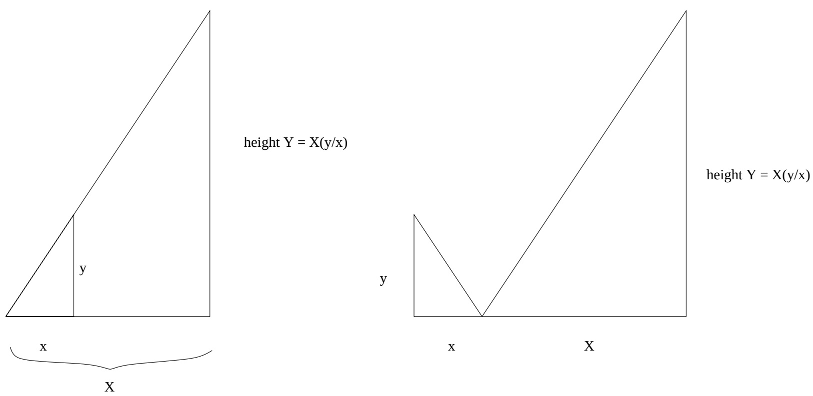


Two objects are *congruent* if they are related by a rigid motion.

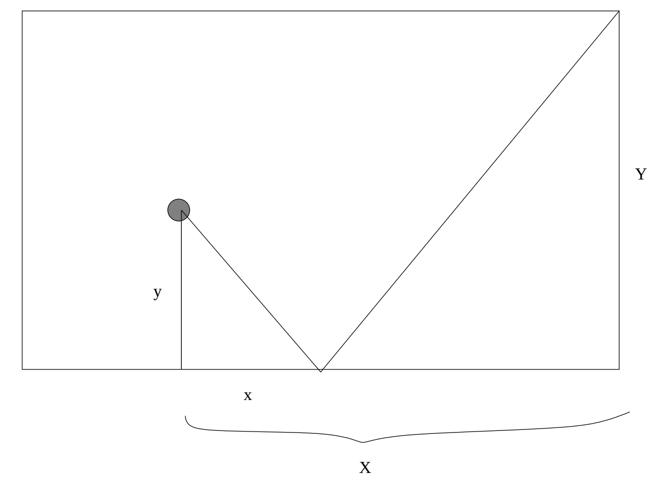
Two triangles are congruent iff they have the same side lengths, or two side lengths and the angle at the intersection point the same, or two angles and the side length between them the same.

Two objects are *similar* if they are related by a rigid motion plus a dilation. Two triangles are similar if they have two angles in common.

Typical application of similarity is determining the heights of buildings etc. Two methods for doing this are:



Similar questions also appear in the games of billiards/pool. For example:



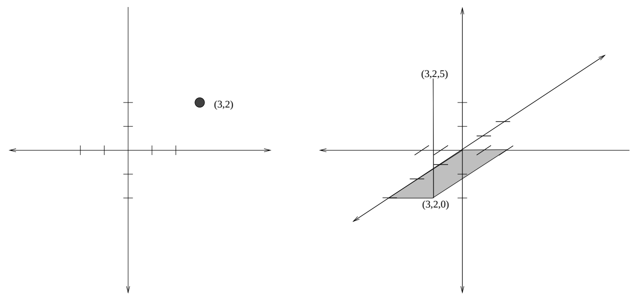
If X = 11 and y = 4 and Y = 9, what must x be in order for the ball to go into the corner pocket

Ans: Y/(X − x) = y/x implies 9/4 = (11 − x)/x = 11/x − 1 so 13/4 = 11/x so x = 44/13

A *symmetry* of an object is a rigid motion that leaves it invariants.

A *n-fold symmetry* is symmetry around a point consisting of rotations by $360/n$ degrees. For example, squares have 4-fold while equilateral triangles have 3-fold symmetry.

Similarly a point in space, equipped with an origin and three coordinate axes, can be described by a triple of coordinates (x, y, z). To draw the point (x, y, z) move x units along the x axis, y units along the y axis, and z units along the z axis. It follows from the Pythagorean theorem, as we will see later, that the distance between two points is the square root of the sum of the squares of the differences between coordinates.



Problem: Find the distance between the points $(1, 2, 3)$ and $(0, 1, 1)$.

Answer: distance is $\sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$.

Problem: Find the distance between the opposite corners of a room of dimension 10 ft by 10 ft by 20 ft.

Answer: distance is $\sqrt{10^2 + 10^2 + 20^2 ft^2} = \sqrt{600} ft$.

5. Lengths, Areas, Volumes

The *length* resp. *area* resp. *volume* of an object is the number of (possibly fractional) units of length resp. area resp. volume needed to cover the object.

We denote by $V(O)$ the length resp. area resp. volume of O for one resp. two resp. three-dimensional objects.

These quantities satisfy the following relations:

(a) if O_1 and O_2 are two objects that do no overlap, then $V(O_1 \cup O_2) = V(O_1) + V(O_2)$.

(b) If O_2 is related to O_1 by a translation, rotation, or reflection, then $V(O_2) = V(O_1)$

(c) If O_2 is related to O_1 by a scaling by a factor s, then $V(O_2) = s^d V(O_1)$.

For example, doubling the size doubles the length, multiplies the areas by four, and multiplies volumes by eight.

Problem: A recipe for apple pie calls for 6 regular size apples. You have a bag of apples whose length, height, and width are half those of regular apples. How many do you need?

Answer: Your smaller apples are related to the regular apples by a scale factor of 1/2. So the volume of a small apple is 1/8 that of a regular apple. So you need 6 times 8 equals 48 smaller apples to make up the same volume.

Problem: In order to tile your basement, you would need 500 square tiles with side lengths 1 ft. Suppose you have, instead, tiles with side lengths 1/2. How many of the smaller tiles do you need?

Answer: the smaller tiles are related to the regular tiles by a scale factor of 1/2, so their area is 1/4 that of the regular tiles. So you need $500 \times 4 = 2000$ of the smaller tiles.

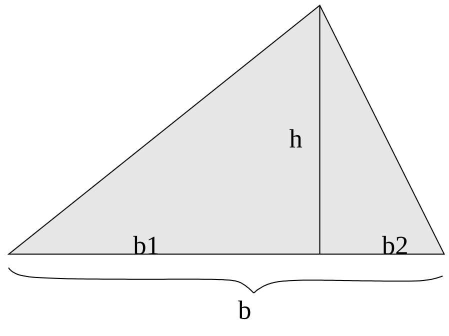
In the case of the length of a curved object, it is defined as the limit of the *approximate polyhedral path*, as the size of the segments in the polyedral path get smaller and smaller.

In the case of the area of a curved object, it is defined as the limit of the *approximate polyhedral object*, as the size of the polygons in the approximate object get smaller and smaller.

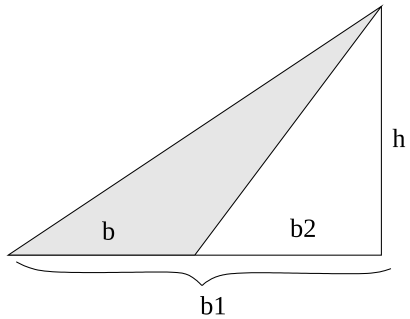
Some formulas for areas of triangles: Three formulas for the area of a triangle are

(a) base times height over 2.

Proof: for a right triangle, it's easy to double it to get a rectangle. Any triangle with a given base can be written either as a sum of two right triangles each with part of the base, or a difference of two triangles sharing part of the same base:



b = b1 + b2



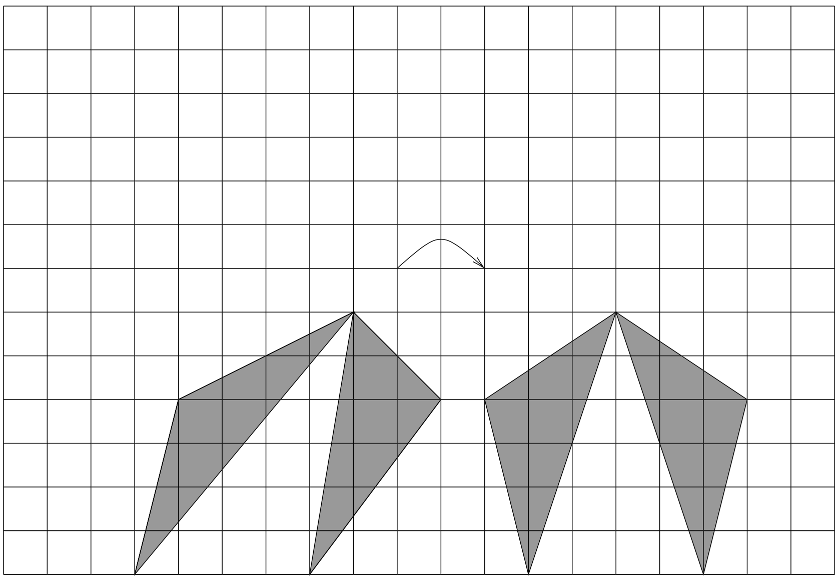
b = b1 − b2

(b) $\pm(ad - bc)/2$ where one vertex is at $(0, 0)$ and the other vertices are at (a, b) and (c, d).

(c) $\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}/4$ where a, b, c are the side lengths. Use formula (c) to find the area of a triangle with side lengths $2, 4, 5$.

The area of a parallogram with base b and height h is bh. In the case that one vertex is over the base, this is easy to see by "cut and paste" of a single triangle. The general case can be seen by "wrapping" the parallogram into a rectangle with the same base and height.

A *shear* with direction d and constant c is a motion of the plane which moves each point in the direction d by a distance equal to c times the height of the point over the line. Shears do not preserve distance, but they do preserve area of a figure. To see this, break the figure up into squares with one side parallel to the direction of the shear. Then the shear transforms each of these squares into parallelograms with the same base and height, hence the same area. Since the sheared figure has area given by the sum of the areas of the parallelograms, it has the same area as the original figure.



Problem: Find the shear of the first figure in the horizontal direction so that the top point is exactly half−way between the two bottom points. Do the triangles have the same area.

Answer: The top point is three to the right of the point half−way between the bottom points, so it needs to be moved three units to the left. Its height is 6, so the shear constant is 1/2. The two points that have height four should then be moved (1/2)4 = 2 to the left, giving the figure above. All the triangles have the same area, since shears preserve area and the two triangles on the irght are reflections of each other.

The two major systems for measurement are the *metric system* (used by the e.g. English) and the *English system* used by the Americans.)

Conversion Factors for Units of Length 1 inch = 2.54 centimeters 1 foot = 0.305 meter 1 yard = 0.914 meter 1 mile = 1.609 kilometers 1 nautical mile = 1.852 kilometers 1 centimeter = 0.39 inch 1 meter = 39.37 inches = 3.28 feet = 1.094 yards 1 kilometer = 0.62 mil American and British Units of Length 1 inch (in.) = 1/36 yard = 1/12 foot 1 foot (ft) = 1/3 yard 1 yard (yd; basic unit of length 1 rod (rd) = 5 1/2 yards 1 furlong (fur.) = 220 yards = 1/8 mile 1 mile (mi) = 1,760 yards = 5,280 feet 1 fathom (fath) = 6 feet 1 nautical mile = 6,076.1 feet Metric Units of Length 1 millimeter (mm) = 1/1,000 meter 1 centimeter (cm)

= 1/100 meter 1 decimeter (dm) = 1/10 meter 1 meter (m; basic unit of length) 1 dekameter (dkm) = 10 meters 1 kilometer (km) = 1000 meters

Two methods for unit conversion are *substitution* and *ratios*.

Substitution means for example $50\ inches^2 = 50((1/12)ft)^2 = 50/144\ ft^2$.

Here are some more examples: (Remember that square units get multiplied by the conversion factor twice and cubic units get multiplied three times!)

Problem: Convert 80 square miles to square kilometers. Answer: $80\ miles^2 = 80(1.6km)^2 = 80(1.6)^2 km^2 = 80(2.56)km^2 = 204.8km^2$.

Problem: Convert 10 cubic feet to cubic meters. Answer $10\ ft^3 = 10\ (1/3.2m)^3 = 10/(3.2)^3 m^3$ which is about $10/32m^3$ or about $3m^3$.

Ratios means for example $50\ inches^2 = 50\ inches^2(1ft/12inches)^2 = 50/144\ ft^2$.

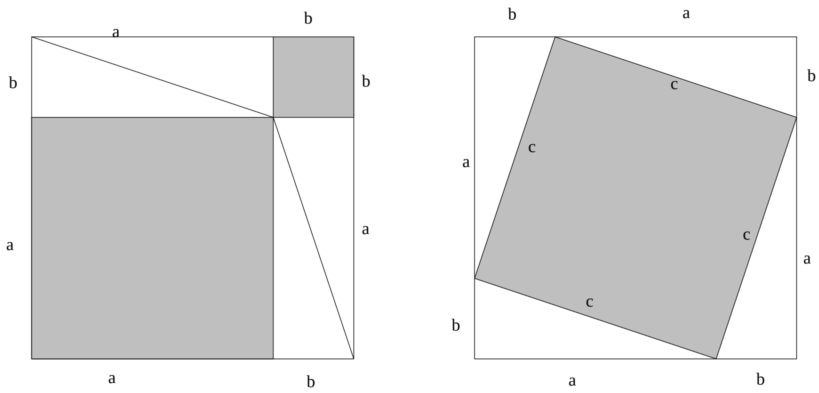
Problem: Convert $10mph$ into m/s using ratios. Answer: (approximate) $10mph = 10miles/hr = 10miles/hr(1600meters/mile)(1hr/60min)(1min/60s) = 16000/3600m/s$ which is about $4.7m/s$.

Note that Celsius to Fahrenheit is a *shifted conversion*: temperature in F is $T_F = 32 + (9/5)T_C$. But a shift of S_C in Celsius produces a shift of $S_F = (9/5)S_C$ in Fahrenheit.

6. Pythagorean Theorem

The *Pythagorean theorem* says that a right triangle with lengths a, b and hypotenuse c satisfies $a^2 + b^2 = c^2$.

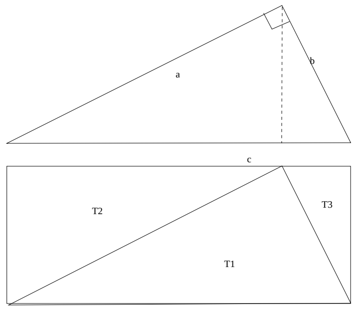
The traditional proof by picture goes like this: Consider the two squares with edge lengths $a + b$:



a^2 + b^2 + 2ab = c^2 + 4(ab/2)

cancel 2ab on both sides to get a^2 + b^2 = c^2

This proof is a little confusing because, after all, where does the picture come from? A better explanation uses the following simpler picture and a little bit about similarity:



Triangles T1,T2,T3 all share two angles, so they are all similar, so they have the same ratio r = height/base.

So Area(T2) = Area(T2) + Area(T3) = (a/c)^2 Area(T1) + (b/c)^2 Area(T2) implies

a^2 + b^2 = c^2

Using the Pythagorean theorem twice one gets

Theorem: the distance between two points is the square root of the sum of the squares of the distances in the three dimensions: if the points have coordinates (x_1, y_1, z_1) and (x_2, y_2, x_2) then

$$dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof: The points (x_1, y_1, z_1) and (x_2, y_2, z_1) are connected by the hypotenuse of a right triangle with side lengths $(x_2 - x_1)$ and $(y_2 - y_1)$, and so the distance between them is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This hypotenuse is the adjacent side of a right triangle whose hypotenuse connects (x_1, y_1, z_1) and (x_2, y_2, z_2), with side lengths $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and $(z_2 - z_1)$. Applying the Pythagorean theorem again gives the result.

For example, if a room is $20ft \times 20ft \times 10ft$ the the distance from one bottom corner to the opposite top corner is

$$dist = \sqrt{(20ft)^2 + (20ft^2) + (10ft)^2} = \sqrt{900ft^2} = 30ft.$$

Problem: What is the height of an Egyptian pyramid with equilateral triangles of edge lengths 2 as faces?

Answer: if the vertices on the base are $(1, 1, 0), (1, -1, 0), (-1, -1, 0), (-1, 1, 0)$ and the vertex at the top is $(0, 0, z)$, then the height is z^2 and the distance to the top is $\sqrt{1^2 + 1^2 + z^2} = 2$. So $z = \sqrt{2}$.

Example: Suppose an Egyptian pyramid has side lengths 400 ft and height 200 ft. What are the areas of the sides?

Answer: The distance from the vertex of the pyramid to the base of each triangular side is the hypotenuse of a triangle with opposite and adjacent sides $200ft$, $200ft$. So the "height" of each triangular side is $\sqrt{200^2 + 200^2} = \sqrt{80000}$. So the area of each triangular side is $1/2(400ft)\sqrt{80000}ft$, or about 75000 ft^2.

If an object is travelling at *constant speed*, then the distance travelled is speed x time. If the object is travelling at variable speed, then the *average speed* is the distance divided by the time.

For example, if a fly travels from one corner of a $20ft \times 20ft \times 10ft$ room to the other in 10 seconds, then its speed is $30ft/10sec = 3ft/sec$.

The total distance travelled is the area under the graph of the velocity.

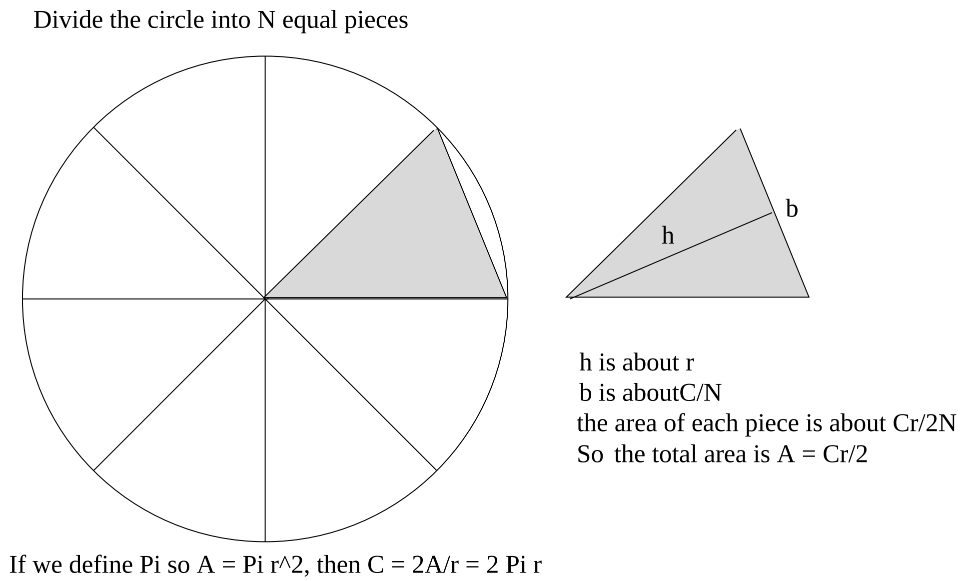
7. Circles, spheres, cones

π is by definition the area of the unit circle.

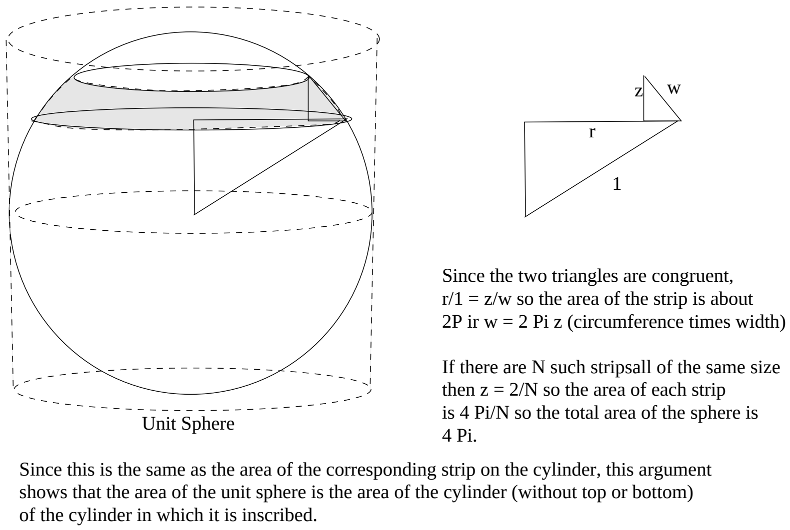
Some formulas for circles and spheres are:

πr^2 area of the circle. (This is basically by definition of π and the behavior of area under rescaling.)

$2\pi r$ circumference of the circle.



$4\pi r^2$ surface area of the sphere. Archimedes computation of this goes as follows:



ARCHMIEDES COMPUTATION OF THE AREA OF THE SPHERE

$(4/3)\pi r^3$ volume of the sphere.