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Constructing and Classifying Fully Irreducible Outer Automorphisms of Free Groups

NOTE: A thesis/preprint draft is available at: www.math.rutgers.edu/~cpfaff/Thesis.pdf

1 Introduction

The main theorem motivating my research comes from mapping class group theory. The *mapping class group* $MCG(S)$ of a compact surface S is the group of homotopy classes of homeomorphisms $h: S \rightarrow S$. The most common mapping classes are *pseudo-Anosov mapping classes*. Because of their fundamental importance in topology and geometry, both mapping class groups and pseudo-Anosov mapping classes have been objects of extensive research. The list of singularity indices associated to a pseudo-Anosov class is an important invariant of the class. Howard Masur and John Smillie proved precisely which lists of singularity indices arise from pseudo-Anosov classes [MS93]. (Some representative of a pseudo-Anosov class expands and contracts a pair of transverse measured foliations on the surface. Each foliation singularity has an index.) The main goal of my thesis research was to determine and prove the most appropriate analogue for outer automorphism groups of free groups. For a free group of rank r , F_r , the *outer automorphism group*, $Out(F_r)$, consists of equivalence classes of automorphisms $\Phi: F_r \rightarrow F_r$, where automorphisms differing by an inner automorphism (map Φ_b defined by $\Phi_b(a) = b^{-1}ab$ for all $a \in F_r$) are equivalent.

Questions 2.6 and 2.7 below indicate conjectures giving two possibilities for an analogue theorem. We have not fully answered either of these questions yet, but give a partial answer to the latter in Theorem 2.13. Theorem 2.13 is our main result thus far and the methods constructed for its proof can be used to continue expanding the result. Proposition 2.3 and Lemma 2.12, both used in the proof of Theorem 2.13, also have independent significance.

Outer automorphisms can be described geometrically as follows. Let R_r be the r -petaled rose (graph with r edges and a single vertex v), with v as its basepoint. Given a base-pointed graph Γ with $\pi_1(\Gamma, *) \cong F_r$, we can assign to Γ a *marking* via a homotopy equivalence $m: R_r \rightarrow \Gamma$ where $m(v) = *$. We call such a graph Γ , together with m , a *marked graph*. Each $\phi \in Out(F_r)$ can be represented geometrically by a homotopy equivalence $g: \Gamma \rightarrow \Gamma$ of a marked graph (g determines an element of $Out(\pi_1(\Gamma, *)) \cong Out(F_r)$). Thurston defined such a homotopy equivalence to be a *train track map* if the restriction of g^k to the interior of each edge of Γ is locally injective for all $k > 0$. When g induces $\phi \in Out(F_r)$, we say that g is a *train track representative* for ϕ [BH92]. Train track representatives of outer automorphisms are in many ways the most natural representatives to work with and [BH92] gives an algorithm for finding a train track representative of any irreducible $\phi \in Out(F_r)$. In this document we assume that all representatives are train track representatives.

There is potential for an $Out(F_r)$ -analogue to the Masur-Smillie theorem because of the deep connections between the $Out(F_r)$ and the $MCG(S)$. In the case where $r = 2$, $Out(F_2) \cong Out(\Pi_1(\Sigma_{1,1})) \cong MCG(\Sigma_{1,1})$, where $\Sigma_{1,1}$ denotes a genus-1, once-punctured torus. Furthermore, elements $\phi \in Out(F_2)$ are induced by homeomorphisms of $\Sigma_{1,1}$ and *fully irreducible outer automorphisms* (outer automorphisms ϕ such that no representative of any ϕ^k has an invariant subgraph) are induced by pseudo-Anosov homeomorphisms. While we do not have such an exact correspondence for $r > 2$, there are still strong similarities between all of the $Out(F_r)$ and $MCG(S)$, as well as between the fully irreducible $\phi \in Out(F_r)$ and pseudo-Anosov $\psi \in MCG(S)$. In fact, some $\phi \in Out(F_r)$ with $r > 2$ are even still induced by homeomorphisms of compact surfaces with boundary (such ϕ are called *geometric*). Nongeometric fully irreducible outer automorphisms are either “ageometric” or “parageometric” (as defined by M. Lustig). Our work focuses on the most common class of nongeometric fully irreducible outer automorphisms, the ageometric outer automorphisms.

There is a large group of mathematicians exploring the parallel properties between the $Out(F_r)$ and $MCG(S)$. They have made significant progress in this direction. We use some of their definitions and machinery (including the definitions of singularities, indices, and ideal Whitehead graphs for outer automorphisms defined in [GJLL98] and [HM11]) to understand an appropriate $Out(F_r)$ -analogue to the Masur-Smillie mapping class group theorem. The definitions are given below. As also stated below, we have the examples necessary for an $Out(F_r)$ -analogue in rank three but, since the methods generating the examples are less generalizable, we have postponed work on justifying and publishing them.

There is a more appropriate conjectured theorem in the $Out(F_r)$ setting than that recording possible singularity index lists since, for the $Out(F_r)$, an ideal Whitehead graph gives an even finer invariant than a singularity index list. My thesis work thus actually focuses on determining which potential ideal Whitehead graphs arise from fully irreducible $\phi \in Out(F_r)$. We have shown that precisely eighteen of the twenty-one connected graphs with five vertices are ideal Whitehead graphs for ageometric, fully irreducible $\phi \in Out(F_r)$ (Theorem 2.13) and that the same holds for the complete $(2r - 1)$ -vertex graph for arbitrary $r \geq 3$ (Theorem 2.14). On the other hand, we showed in Lemma 2.12 that a certain type of graph cannot arise from a fully irreducible $\phi \in Out(F_r)$. We are in the process of extending and revising the machinery we developed in hopes of expanding our partial answer to the ageometric case of Question 2.7. A portion of this machinery is included in my online draft.

The remainder of my thesis work includes the development of several methods (not described here) for identifying complications called periodic Nielsen paths and a proof of a criterion identifying representatives of fully irreducible $\phi \in Out(F_r)$ (Lemma 2.12). Both of these results are necessary for the proof of Theorem 2.13, but have independent significance.

Finally, we have created numerous examples of train track representatives with varying properties, many of which are included in my thesis. Constructing and identifying fully irreducible outer automorphisms is of great interest to mathematicians. There are several recent papers to this affect (including [CP10]).

2 Definitions and Theorems

Definition 2.1. ([BH92], [BFH97]) A $\phi \in \text{Out}(F_r)$ is *reducible* if there are proper free factors F_1, \dots, F_k of F_r such that ϕ permutes the conjugacy classes of the F_i and $F_1 * \dots * F_k$ is a free factor of F_r . Otherwise, ϕ is *irreducible*. If every power of ϕ is irreducible, then ϕ is *fully irreducible*. A train track representative g is *irreducible* if it has no invariant subgraph.

Definition 2.2. ([BH92]) A path ρ between fixed points $x, y \in \Gamma$ is a *periodic Nielsen Path (PNP)* if, for some k , $g^k(\rho) \simeq \rho$ rel endpoints. ρ is a *Nielsen Path (NP)* if $k = 1$.

Unless otherwise specified, Γ will be a rose and $g: \Gamma \rightarrow \Gamma$ a PNP-free train track representative of an ageometric, fully irreducible $\phi \in \text{Out}(F_r)$ having $2r - 1$ “fixed directions,” decomposable as a composition of *proper full folds* of roses (maps $e_i \mapsto e_i e_j$, where e_i and e_j are edges in Γ). The following proposition proves the existence of such representatives, which we call *ideally decomposed*.

Proposition 2.3. (Pfaﬀ, '11) Suppose $\phi \in \text{Out}(F_r)$ is an ageometric, fully irreducible outer automorphism such that $IW(\phi)$ is a connected graph with $2r - 1$ vertices. Then there exists a PNP-free, rotationless representative of a power $\psi = \phi^R$ of ϕ on the rose. Furthermore, the representative can be decomposed as a sequence of proper full folds between roses.

(The point of ψ being rotationless is that the representative fixes the periodic directions.)

We will let $\mathcal{E}^+(\Gamma) = \{E_1, E_2, \dots, E_r\}$ be the set of edges of Γ with a prescribed orientation and $\mathcal{E}(\Gamma) = \{E_1, \bar{E}_1, E_2, \bar{E}_2, \dots, E_r, \bar{E}_r\} = \{e_1, e_2, \dots, e_{2r-1}, e_{2r}\}$.

In order to define singularity indices (the weaker outer automorphism invariant), we first give a special case definition for an ideal Whitehead graph (the finer outer automorphism invariant). For this recall the following:

Definition 2.4. ([BH92]) A *Direction* at a point $x \in \Gamma$ is a germ of edge-segments emanating from v . A direction d is *periodic* if $g^k(d) = d$ for some $k > 0$ and *fixed* if $k = 1$. The map of directions induced by g is denoted Dg . For an edge $e \in \mathcal{E}(\Gamma)$, \bar{e} denotes e with the opposite orientation, $d = D_0(e)$ is the germ of an initial segment of e , and $\bar{d} = D_0(\bar{e})$.

([HM11]) For a fixed $x \in \Gamma$, the *local Whitehead graph at x* , $LW(x; \Gamma)$, is the graph whose vertices correspond to the directions at x and edges connect pairs $\{d_i, d_j\}$ when there exists an $E \in \mathcal{E}(\Gamma)$ and $k > 0$ such that $g^k(E)$ takes the turn $\{d_i, d_j\}$, i.e., contains either $\bar{e}_i e_j$ or $\bar{e}_j e_i$. The *local Stable Whitehead graph at x* , $SW(x; \Gamma)$, is the subgraph obtained by restricting to the periodic direction vertices and edges between. If g is ideally decomposable (including PNP-free), then $SW(v; \Gamma) \cong IW(\phi)$, the *ideal Whitehead graph of ϕ* .

Definition 2.5. ([GJLL98], [HM11]) For a fully irreducible ϕ and component C of $IW(\phi)$, define the *index* $i(C) = 1 - \frac{\#\text{vertices}(C)}{2}$. The *index sum of ϕ* is $i(\phi) = \sum_C i(C)$, taken over the components of $IW(\phi)$. The *index type of ϕ* is the list of indices of components of $IW(\phi)$, written in nondecreasing order. (Note: $IW(\phi)$ has only one component in our case).

[GJLL98] gives the index inequality $i(\phi) \geq 1 - r$ for fully irreducible $\phi \in \text{Out}(F_r)$. The first goal of my research was to search for an answer to the following question in [HM11]:

Question 2.6. Which possible index types, satisfying the index inequality $i(\phi) \geq 1 - r$, are achieved by nongeometric, fully irreducible $\phi \in \text{Out}(F_r)$?

When $r = 3$, the index inequality allows precisely $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2}, -1)$, $(-\frac{1}{2}, -\frac{3}{2})$, $(-1, -1)$, $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2})$, and (-1) [HM11]. We found examples of fully irreducible $\phi \in \text{Out}(F_r)$ achieving each of these index types and have ideas for proving their legitimacy, but currently prioritize work using our other constructions, as the methods used to find the index list examples do not easily generalize to $r \geq 3$ (unlike our methods used to construct examples achieving ideal Whitehead graphs, which are valid in every rank).

As stated above, a finer invariant than index type is the isomorphism type of $IW(\phi)$. In the analogous pseudo-Anosov situation, the ideal Whitehead graph isomorphism type is always a circle. In contrast, there are nongeometric, fully irreducible $\phi \in \text{Out}(F_r)$ with connected $IW(\phi)$ not homeomorphic to a circle. With this in mind, our refined research direction was an answer to (Lemma 2.8, Theorem 2.13, and Theorem 2.14 give partial answers):

Question 2.7. Which isomorphism types of graphs occur as $IW(\phi)$ for some fully irreducible $\phi \in \text{Out}(F_r)$?

We first proved that there are connected graphs with $2r - 1$ vertices that do not arise as $IW(\phi)$ for ageometric, fully irreducible $\phi \in \text{Out}(F_r)$:

Lemma 2.8. (Pfaff, '10) *There exists no ageometric, fully irreducible $\phi \in \text{Out}(F_r)$ such that $IW(\phi)$ is the graph consisting of $2r - 2$ edges adjoined at a single vertex.*

To generalize this lemma we invented “Lamination Train Track (LTT) Structures.” LTT Structures serve as nodes in “Admissible Map Diagrams” (used to “construct” ideally decomposed representatives), their “birecurrency” is necessary for full irreducibility, and they contain leaves of laminations as locally smoothly embedded lines.

Definition 2.9. (Pfaff, '10) Let $g: \Gamma \rightarrow \Gamma$ be an ideally decomposable train track representative with notation as given directly below Proposition 2.3.

We define the *Colored local Whitehead graph at the vertex $v \in \Gamma$* , $CW(g; v)$, as $LW(g; v)$ with edges of the subgraph $SW(g; v)$ colored purple, edges of $LW(g; v) - SW(g; v)$ colored red, nonperiodic direction vertices colored red, and periodic direction vertices colored purple.

Obtain Γ_N from Γ by removing a contractible neighborhood, $N(v)$, of v and adding vertices d_i and \overline{d}_i at the corresponding boundary points of the partial edges $E_i - (N(v) \cap E_i)$. Identify each vertex labeled d_i in Γ_N with the vertex labeled d_i in $CW(g; v)$ to form the *Lamination Train Track (LTT) Structure $G_T(g)$* for g . Nonperiodic direction vertices are colored red, edges of Γ_N remain black, and periodic direction vertices remain purple.

An LTT structure $G_T(g)$ is given a *Smooth Structure* via a partition of the edges at each $G_T(g)$ -vertex into: the set of black edges of $G_T(g)$, \mathcal{E}_b , and the set of colored edges, \mathcal{E}_c .

Note that $G_T(g)$ can be built from $CW(g; \Gamma)$ by adding a black edge for each vertex pair $\{d_i, \overline{d}_i\}$. Also note that each edge image path $g(e_i) = e_{j_1} \dots e_{j_k}$ determines a path in $G_T(g)$ transversing the black edge corresponding to $\{d_{j_1}, \overline{d}_{j_1}\}$, then colored edge corresponding to

$\{\overline{d_{j_1}}, d_{j_2}\}$, black edge corresponding to $\{d_{j_2}, \overline{d_{j_2}}\}$, etc, until ending with the black edge corresponding to $\{d_{j_k}, \overline{d_{j_k}}\}$. This observation relates to an important property of LTT structures: they contain leaves of the expanding lamination as locally smoothly embedded lines.

Definition 2.10. (Pfaff, '10) A LTT structure G is *birecurrent* if there exists a locally smooth embedding of \mathbb{R} into G that crosses each edge of G infinitely often as $\mathbb{R} \rightarrow \infty$ and as $\mathbb{R} \rightarrow -\infty$.

We are now able to state our generalization of Lemma 2.8:

Lemma 2.11. (Pfaff, '10) *The LTT structure for a train track representative of a fully irreducible outer automorphism is birecurrent.*

This birecurrency condition is interesting and useful, but insufficient: We found connected five-vertex graphs not belonging to any ageometric, fully irreducible $\phi \in \text{Out}(F_r)$, but having birecurrent LTT structures. We still aspire to fully characterize all obtainable ideal Whitehead graphs. In the meantime, we developed machinery for constructing ideally decomposed representatives, used this machinery to construct representatives with many different ideal Whitehead graphs (including all possible connected five-vertex graphs, see Theorem 2.13), and continue to use it to construct examples and prove others nonexistent.

In combination with our construction methods we had to develop means to check and ensure the following necessary conditions:

(I) Our representatives are PNP-free.

(II) Our representatives represent fully irreducible outer automorphisms.

To identify all PNPs or show that they do not exist, we developed two procedures. One can be found in my thesis and the other involves a computer program written by Andrew Baxter of Rutgers. For (II), with the help of M. Handel (and the *Completely Split Train Track* machinery of [FH09]), we proved the following ‘‘Folk lemma’’:

Lemma 2.12. (Pfaff, '11) *(The Full Irreducibility Criterion) Let $g: \Gamma \rightarrow \Gamma$ be an irreducible train track representative of $\phi \in \text{Out}(F_r)$. Suppose that g has no PNPs, the transition matrix for g is Perron-Frobenius, and all $LW(g; x)$ for g are connected. Then ϕ is fully irreducible.*

An advertisement for our methods was their efficiency in proving the following:

Theorem 2.13. (Pfaff, '11) *Exactly eighteen of the twenty-one connected graphs with five vertices are ideal Whitehead graphs for ageometric, fully irreducible $\phi \in \text{Out}(F_3)$.*

One unachieved graphs has no birecurrent LTT structures (the graph of four edges adjoined at a vertex, as in Lemma 2.8). While the other graphs have several birecurrent LTT structures, they are few among all LTT structures with those purple subgraphs. This gives a reason why some graphs are never $IW(\phi)$ for a ageometric, fully irreducible $\phi \in \text{Out}(F_r)$.

Another use of the methods was:

Theorem 2.14. (Pfaff, '11) *For each $r \geq 3$ there exists an ageometric, fully irreducible $\phi \in \text{Out}(F_r)$ such that $IW(\phi)$ is the complete graph on $2r - 1$ vertices.*

It should be noted that our methods and proofs hold in all ranks. On the other hand, while many of our constructions and definitions work for graphs other than roses, PNP-related obstacles, make it unclear how to extend our methods to these graphs.

3 Future Research

We would like to more fully answer Question 2.6 and Question 2.7, as well as find shorter methods for constructing representatives.

We are still in the process of exhausting the usefulness of our construction methods in the direction of more fully answering Question 2.7. However, in order to obtain index types with more than one entry, we either need to generalize our constructions to graphs other than roses or find representatives on roses with PNPs that can be “collapsed” to form PNP-free representatives. In rank-three, we found it much simpler to use the method of “collapsing” PNPs. Expanding the theory behind “collapsing” PNPs and generally handling their presence could also open new research directions. On the other hand, generalizations of our definitions may make this unnecessary.

Since the birecurrency condition is insufficient, we would still like to find a condition completely answering Question 2.7. The fact that leaves of laminations for a train track representative give smooth paths in the LTT structure relates properties of LTT structures to those of laminations, something much better understood. We thus may be able to use laminations to find a better condition. We also have other methods for answering Question 2.7. For example, we may be able to construct ideal Whitehead graphs in “blocks” from representatives already constructed. Additionally, as we did in Theorem 2.14, we could create rank-independent procedures for constructing certain types of graphs in any rank.

Many of our desired maps give paths in two structures we defined, AM Diagrams and IWG Building Graphs. Consequently, we could improve our construction and identification methods by better understanding paths in these structures. Alternatively, we could shorten procedures by refining the structures themselves.

Finally, there are questions to answer, such as one posed by J. Behrstock, about the limit (as the rank approaches infinity) of the proportion of achievable to unachievable potential ideal Whitehead graphs.

4 Broader Research Interests

In addition to $Out(F_n)$ theory, I have devoted significant time to studying mapping class groups and Teichmueller theory. I have attended a multitude of conferences on these topics; attended (and participated in) multiple seminars on these topics; attended (with my advisor) the MSRI semester on geometric group theory, Teichmueller theory, and Kleinian groups; and was even asked to help edit a chapter of “A Primer on Mapping Class Groups” by B. Farb and D. Margalit. I have also devoted significant time to learning about other areas of geometric group theory, topology, and geometry. There are many areas of geometric group theory and low-dimensional topology, in particular, that I am very interested in collaborating on research in. For example, those areas involving Bass-Serre theory, right-angled Artin groups, coarse geometry, and subjects related to mapping class groups (such as those involving homeomorphisms of boundaries, laminations, or foliations).

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