Analytic Methods for Obstruction to Integrability in Discrete Dynamical Systems

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Abstract

A unique analytic continuation result is proven for solutions of a relatively general class of difference equations by using techniques of generalized Borel summability.

This continuation allows for Painlevé property methods to be extended to difference equations.

It is shown that the Painlevé property (PP) induces, under relatively general assumptions, a dichotomy within first-order difference equations: all equations with PP can be solved in closed form; on the contrary, absence of PP implies, under some further assumptions, that the local conserved quantities are strictly local in the sense that they develop singularity barriers on the boundary of some compact set.

The technique produces analytic formulas to describe fractal sets originating in polynomial iterations. © 2004 Wiley Periodicals, Inc.

1 Introduction and Main Results

Solvability of difference equations as well as chaotic behavior have stimulated extensive research. For differential equations the Painlevé test, which consists in checking whether all solutions of a given equation are free of movable nonisolated singularities, provides a convenient and effective tool in detecting integrable cases (see Section 2).

A difficulty in applying Painlevé's methods to difference equations resides in extending the solutions, which are defined on a discrete set, to the complex plane of the independent variable in a natural and effective fashion, when, in the interesting cases, there is no explicit formula for them. A number of alternative approaches, but no genuine analogue of the Painlevé test, have been proposed; see [1, 9, 19, 21] (a comparative discussion of the various approaches is presented in [1]).

The present paper proposes a natural way, based on generalized Borel summability, to extend the solutions in the complex plane (Theorem 1.1 below), allowing for a definition of a discrete Painlevé test. Subsequent analysis shows that the test

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is sharp in a class of first-order difference equations: those passing the test are explicitly solvable (Theorem 1.5), while polynomial equations failing the test exhibit chaotic behavior, and their local conserved quantities (see Section 1.8) develop barriers of singularities along fractal sets (Theorem 1.8).

The approach also allows for a detailed study of analytic properties near these singularity barriers as well as finding rapidly convergent series representing the corresponding fractal curves (Theorem 1.10).

1.1 Setting

We consider difference systems of equations that can be brought to the form

$$\mathbf{x}(n+1) = \hat{\Lambda}\left(I + \frac{1}{n}\hat{A}\right)\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{1.1}$$

where $\hat{\Lambda}$ and $\hat{A}$ are constant-coefficient matrices, $\mathbf{g}$ is convergently given for small $\mathbf{x}$ by

$$\mathbf{g}(n, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{g}_{\mathbf{k}}(n)\mathbf{x}^{\mathbf{k}} \tag{1.2}$$

with $\mathbf{g}_{\mathbf{k}}(n)$ analytic in n at infinity, and

$$\mathbf{g}_{\mathbf{k}}(n) = O(n^{-2}) \quad \text{as } n \to \infty \text{ if } \sum_{j=1}^{m} k_j \le 1 \tag{1.3}$$

under nonresonance conditions: Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\mathbf{a} = (a_1, \ldots, a_n)$ where $e^{-\mu_k}$ are the eigenvalues of $\hat{\Lambda}$ and the a_k are the eigenvalues of $\hat{A}$. Then the nonresonance condition is

$$(\mathbf{k}\cdot\boldsymbol{\mu} = 0 \bmod 2\pi i \text{ with } \mathbf{k} \in \mathbb{Z}^{m_1}) \Leftrightarrow \mathbf{k} = 0\,. \tag{1.4}$$

We consider the solutions of (1.1) that are small as n becomes large.

1.2 Analyzability: Transseries and Generalized Borel Summability

These concepts were introduced by Écalle in the fundamental work [14]. Analyzability of difference equations was shown in [7, 14]. We give below a brief description of the concepts effectively used in the present paper and refer to [7, 10] for a general theory. An expression of the form

$$\tilde{\mathbf{x}}(t) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu}t} t^{\mathbf{k}\cdot\mathbf{a}} \tilde{\mathbf{x}}_{\mathbf{k}}(t) \tag{1.5}$$

where $\tilde{\mathbf{x}}_{\mathbf{k}}(t)$ are formal power series in powers of t^{-1} is an exponential power series; it is a transseries as $t \to +\infty$ if $\Re(\mu_j) > 0$ for all j with $1 \le j \le m$. Such a transseries is Borel-summable as $t \to +\infty$ if there exist constants $A, \nu > 0$ and a family of functions

$$\begin{gathered}\mathbf{X}_{\mathbf{k}} \text{ analytic in a sectorial neighborhood } \mathcal{S} \text{ of } \mathbb{R}^+ \text{ satisfying}\\ \sup_{p\in\mathcal{S},\mathbf{k}\in\mathbb{N}^m} \left|A^{|\mathbf{k}|}e^{-\nu|p|}\mathbf{X}_{\mathbf{k}}\right| < \infty\end{gathered} \tag{1.6}$$

such that the functions $\mathbf{x}_{\mathbf{k}}$ defined by

$$\mathbf{x}_{\mathbf{k}}(t) = \int_0^\infty e^{-tp}\mathbf{X}_{\mathbf{k}}(p)dp \tag{1.7}$$

are asymptotic to the series $\tilde{\mathbf{x}}_{\mathbf{k}}$, i.e.,

$$\mathbf{x}_{\mathbf{k}}(t) \sim \tilde{\mathbf{x}}_{\mathbf{k}}(t) \quad t \to +\infty\,. \tag{1.8}$$

It is then easy to check that condition (1.6) implies that the sum

$$\mathbf{x}(t) = \sum_{\mathbf{k}\in\mathbb{N}^n0} \mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\mu}t}t^{\mathbf{k}\cdot\mathbf{a}}\mathbf{x}_{\mathbf{k}}(t) \tag{1.9}$$

is convergent in the half-plane $\mathbb{H} = \{t : \Re(t) > t_0\}$ for t_0 large enough. The function $\mathbf{x}$ in (1.9) is by definition the Borel sum of the transseries $\tilde{\mathbf{x}}$ in (1.5). Generalized Borel summability allows for singularities of $\mathbf{X}_{\mathbf{k}}$ of certain types along $\mathbb{R}^+$. The transseries $\tilde{\mathbf{x}}$ is (generalized) Borel-summable in the direction $e^{i\varphi}\mathbb{R}^+$ if $\tilde{\mathbf{x}}(\cdot e^{-i\varphi})$ is (generalized) Borel-summable. (Generalized) Borel summation is known to be an extended isomorphism between transseries and their sums; see [10, 14, 31].

Transseries for Difference Equations

Braaksma [7] showed that the recurrences (1.1) possess l-parameter transseries solutions of the form (1.5) with $t = n$ where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in powers of n^{-1} and $l \le m$ is chosen such that, after reordering the indices, we have $\Re(\mu_j) > 0$ for $1 \le j \le l$.

It is shown in [7] and [25] that these transseries are generalized Borel-summable in any direction and Borel-summable in all except m of them and that

$$\mathbf{x}(n) = \sum_{\mathbf{k}\in\mathbb{N}^l} \mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\mu}n}n^{\mathbf{k}\cdot\mathbf{a}}\mathbf{x}_{\mathbf{k}}(n) \tag{1.10}$$

is a solution of (1.1), if $n > y_0$, t_0 large enough.

1.3 Uniqueness of Continuation from $\mathbb{N}$ to $\mathbb{C}$

The values of $\mathbf{x}$ on the integers uniquely determine $\mathbf{x}$.

THEOREM 1.1 *In the assumptions in Section* 1.1 *and* 1.2*, define the continuation of* $\mathbf{x}_{\mathbf{k}}(n)$ *in the half-plane* $\{t : \Re(t) > t_0\}$ *by* $\mathbf{x}(t)$; *cf.* (1.6)–(1.9).

The following uniqueness property holds: If in the assumptions (1.6)–(1.9) *we have* $\mathbf{x}(n) = 0$ *for all except possibly finitely many* $n \in \mathbb{N}$*, then* $\mathbf{x}(t) = 0$ *for all* $t \in \mathbb{C}$, $\Re(t) > t_0$.

The proof is given in Section 3.1.

1.4 Continuability and Singularities

The function $\mathbf{x}$ is analytic in $\mathbb{H}$ and has, in general, nontrivial singularities in $\mathbb{C} \setminus \mathbb{H}$. The results in [11], extended to difference equations in [7, 8, 25], give constructive methods to determine those singularities that arise near the boundary of $\mathbb{H}$; these form, generically, nearly periodic arrays.

1.5 Integrability

In particular, Painlevé's test of integrability (absence of movable nonisolated singularities) extends then to difference equations.

As in the case of differential equations, fixed singularities are singular points whose location is the same for all solutions; they define a common Riemann surface. Other singularities (i.e., whose location depends on initial data) are called *movable*.

Representation (1.10) and Theorem 1.1 make the following definition natural:

DEFINITION 1.2 We say that a difference equation has the Painlevé property if its solutions are analyzable and their analytic continuations on a Riemann surface *common to all solutions* have only *isolated* singularities.

Note. We follow the usual convention that an isolated singular point of an analytic function f is a point z_0 such that f is analytic in some disk centered at z_0 except perhaps at z_0 itself. Branch points are thus not isolated singularities and neither are singularity barriers; it is worth noting, however, that for differential equations there exist equations sometimes considered integrable (the *Chazy equation*, a third-order nonlinear one, is the simplest known example) whose solutions exhibit singularity barriers.

1.6 First-Order Autonomous Equations

These are equations of the type

$$x_{n+1} = G(x_n) := ax_n + F(x_n)\,. \tag{1.11}$$

Some analyticity assumptions on F are required for our method to apply. We define a class of single-valued functions closed under all algebraic operations and composition (the latter is needed since x_n written in terms of x_0 involves repeated composition).

We need to allow for singular behavior in F, and meromorphic functions are obviously not closed under composition. The following definition formalizes an extension of meromorphic functions, often used informally in the theory of integrability.

DEFINITION 1.3 We define the *mostly analytic functions* to be the class $\mathcal{M}$ of functions analytic in the complement of a closed countable set (which may depend on the function).

LEMMA 1.4

(i) *The class $\mathcal{M}$ is closed under addition, multiplication, and multiplication by scalars, and also under division and composition between* (*nonconstant*) *functions. It includes meromorphic functions.*

(ii) *If $G \in \mathcal{M}$ is not a constant, then the equation $G(x) = y$ has solutions for all large enough y.*

(iii) *The class $\mathcal{M}_0$ of $G \in \mathcal{M}$, with G analytic at zero, $G(0) = 0$, and $0 < |G'(0)| < 1$ is closed under composition.*

In particular, $G^{\circ m} \in \mathcal{M}$ for $m \geq 1$.

PROOF: All properties in (i) are obvious except for closure under composition and division, proven in Section 3.3; (ii) follows from the proof of Lemma A.9; and (iii) is easily shown using (i). □

1.7 Classification of Equations of Type (1.11) with Respect to Integrability

THEOREM 1.5 *Assume $G \in \mathcal{M}$ has a stable fixed point* (*say at zero*) *where it is analytic. Then the difference equation* (1.11) *has the Painlevé property if and only if for some $a, b \in \mathbb{C}$ with $|a| < 1$,*

$$G(z) = \frac{az}{1 + bz} \,. \tag{1.12}$$

The proof is given in Section 3.4.

Remark 1.6. The Painlevé property *is not sensitive to which attracting fixed point of G or its iterates is used in the analysis. This follows from the proposition below.*

Assume p is another attracting fixed point of G and let $G_1(s) = G(p + s) - p$ (G_1 has an attracting fixed point at the origin).

PROPOSITION 1.7 *The difference equation* (1.11) *has the Painlevé property if and only if the difference equation $x_{n+1} = G_1(x_n)$ has the Painlevé property. Furthermore, if G has an iterate $G^{\circ m}$ with an attracting fixed point where the conjugation map extends analytically to $\mathbb{C}$ except for isolated singularities, then the same is true for any attracting fixed point of any iterate $G^{\circ k}$.*

This is shown in Section A.1.

1.8 Failure of Integrability Test and Barriers of Singularities

Conserved quantities are naturally defined as functions $C(x; n)$ with the property

$$C(x_{n+1}; n + 1) = C(x_n; n) \,.$$

We now look at cases without the Painlevé property when G is a polynomial map. We arrive at the striking conclusion that these equations are not solvable in terms of functions extendible to the complex plane or on Riemann surfaces. The conserved quantities will typically develop singularity barriers.

We use, in the formulation of the following theorem, a number of standard notions and results relevant to iterations of rational maps; these are briefly reviewed in the appendix.

THEOREM 1.8 *Assume G is a nonlinear polynomial with an attracting fixed point at the origin. Denote by $\mathcal{K}_p$ the maximal connected component of the origin in the Fatou set of G.* (*It follows that $\mathcal{K}_p$ is an open, bounded, and simply connected set*).

Then the domain of analyticity of Q (*see* (3.26)) *is $\mathcal{K}_p$, and $\partial\mathcal{K}_p$ is a singularity barrier of Q.*

This theorem is proven in Section 3.5.

1.9 Example: The Logistic Map

The discrete logistic map is defined by

$$x_{n+1} = ax_n(1 - x_n)\,. \tag{1.13}$$

The following result was proven by the authors in [12]:

PROPOSITION 1.9 *The recurrence* (1.13) *has the Painlevé property noted in Definition* 1.2 *if and only if $a \in \{-2, 0, 2, 4\}$* (*in which cases it is explicitly solvable*). *If $a \notin \{-2, 0, 2, 4\}$, then the conserved quantity has barriers of singularities.*

1.10 Application to the Study of Fractal Sets

The techniques also provide detailed information on the Julia sets of iterations of the interval.

THEOREM 1.10 *Consider equation* (1.13) *for $a \in (0, \frac{1}{2})$.*

(i) *There is an analytic function K, satisfying the functional relation*

$$K(z)^2 = aK(z^2)(1 + K(z)) \tag{1.14}$$

that is a conformal map of the open unit disk S_1 onto $\{x^{-1} : x \in \mathrm{ext}(\mathcal{J})\}$ where $\mathcal{J}$ is the Julia set of (1.13).

(ii) *K is Lipschitz-continuous of exponent* $\log_2(2 - a)$ *in $\overline{S_1}$* (*the Lipschitz constant can be determined from the proof*).

(iii) *∂S_1 is a barrier of singularities of K. Near $1 \in \partial S_1$ we have*

$$K(z) = \Phi(\tau\Psi(\ln\tau)) \tag{1.15}$$

where

$$\tau = \tau(z) = \ln(z^{-1})^{\log_2(2-a)}$$

$$\Phi \textit{ is analytic at zero}, \quad \Phi(0) = \frac{a}{1-a}\,, \quad \Phi'(0) = 1\,, \tag{1.16}$$

$$\Psi \textit{ is real analytic and periodic of period } \ln 2.$$

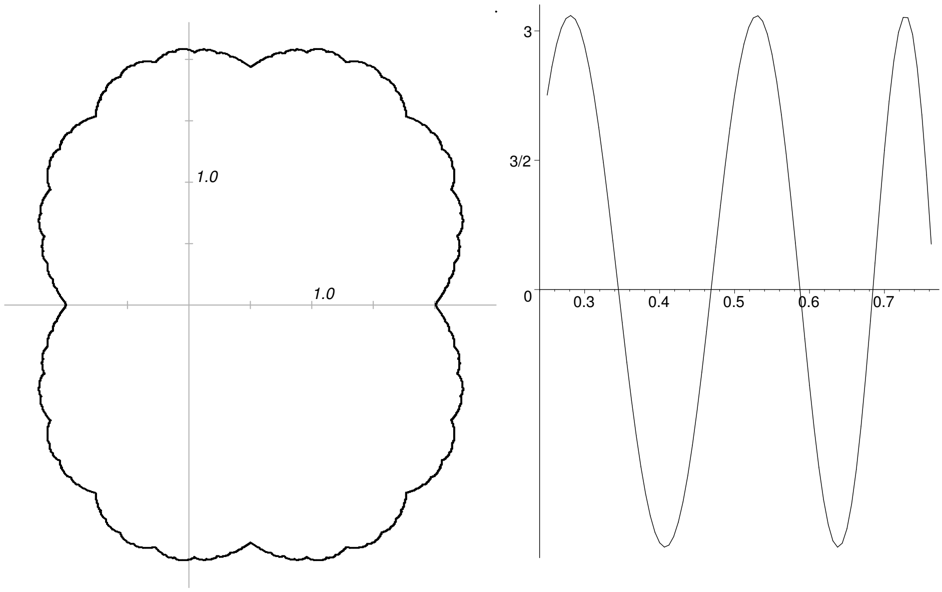

FIGURE 1.1. (a) Julia set for $G = \frac{1}{2}x(1-x)$. The set $\mathcal{K}_p$ is the interior of the curve. (b) The function $10^9(\Psi(\ln\ln z_0) + c)$ for $a = \frac{1}{2}$ and $c = 0.079324389476$ (the plot relies on (5.8), $N = 300$).

With $t = 1 - z$ we have

$$K = \frac{a}{1-a} + \sum_{l\in\mathbb{Z}} \sum_{k,m\in\mathbb{N}} C_{l;k,m} t^{2\pi i l \log_2(2-a)/(\ln 2) + k\log_2(2-a) + m} \tag{1.17}$$

where the series converges (*rapidly*) *if t and $|\arg t|$ are small.*

This theorem is proven in Section 4.

Remark 1.11. The proof of Proposition 5.1 shows that the Lipschitz exponent is optimal. The theorem is valid for any $a < 1$, and the proof is similar.

Remark 1.12. It follows from Theorem 1.10(iii) and (1.14) that every binary rational is a cusp of $\mathcal{J}$ of exterior angle $\pi \log_2(2-a)$; see also Figure 1.1.

2 General Remarks on Integrability

This problem has a long history, and the task of finding *differential* equations solvable in terms of known functions was addressed as early as the works of Leibniz, Riccati, Bernoulli, Euler, Laplace, and Lagrange. "In the 18th century, Euler was defining a function as arising from the application of finitely or infinitely many algebraic operations (addition, multiplication, raising to integer or fractional powers, positive or negative) or analytic operations (differentiation, integration), in one or more variables" [6, chap. 1, p. 1]. It was later found that some linear equations have solutions that, although not explicit by this standard, have "good" *global* properties and can be thought of as defining new functions. To address the question whether nonlinear equations can define new functions, Fuchs had the idea that a crucial feature now known as the *Painlevé property* (PP) is the absence of movable (meaning their position is solution-dependent, cf. Section 1.5) essential singularities, primarily branch points; see [16]. First-order equations were classified with respect to the PP by Fuchs, Briot, and Bouquet, and by Painlevé by 1888, and it was concluded that they give rise to no new functions. Painlevé took this analysis to second order, looking for all equations of the form $u'' = F(u', u, z)$, with F rational in u', algebraic in u, and analytic in z, having the PP [28, 29]. His analysis, revised and completed by Gambier and Fuchs, found some 50 types with this property and succeeded to solve all but six of them in terms of previously known functions. The remaining six types are now known as the Painlevé equations, and their solutions, called the Painlevé transcendents, play a fundamental role in many areas of pure and applied mathematics. Beginning in the 1980s, almost a century after their discovery, these problems were solved, using their striking relation to linear problems,[1] by various methods including the powerful techniques of isomonodromic deformation and reduction to Riemann-Hilbert problems [13, 15, 20].

Sophie Kowalevski searched for cases of the spinning top having the PP. She found a previously unknown integrable case and solved it in terms of hyperelliptic functions. Her work [22, 23] was so outstanding that not only did she receive the 1886 Bordin Prize of the Paris Academy of Sciences, but the associated financial award was almost doubled.

The method pioneered by Kowalevski to identify integrable equations using the Painlevé property is now known as the *Painlevé test*. Part of the power of the Painlevé test stems from the remarkable phenomenon that equations passing it can generally be solved by some method. This phenomenon is not completely understood. At an intuitive level, however, if for example all solutions of an equation are meromorphic, then by solving the equation "backwards," these solutions and their derivatives can be written in terms of the initial conditions. This gives rise to sufficiently many integrals of motion with good regularity properties *globally* in the complex plane.

The Painlevé test has some drawbacks, notably lack of invariance under transformations. To overcome them, [24] introduced the poly-Painlevé test.

3 Proofs

3.1 Proof of Theorem 1.1

Outline

The idea of the proof is to use the convergence of (1.9) and its asymptotic properties to show that all terms $\mathbf{x_k}$ vanish.

We start with some preparatory results.

[1] Some linear problems conducive to Painlevé equations were known already at the beginning of last century. In 1905 Fuchs found a linear isomonodromic problem leading to P_{VI}.

Remark 3.1. If $\mathbf{x_k} \not\equiv 0$ then also $\mathbf{X_k} \not\equiv \mathbf{0}$ (see (1.7)) so for small p we have $\mathbf{X_k} = \sum_{j=L_\mathbf{k}}^{\infty} \mathbf{c}_j p^j$ with $\mathbf{c}_{L_\mathbf{k}} \neq 0$ for some $L_\mathbf{k} \geq 0$. By Watson's lemma [5], for large z in the right half-plane we have

$$\mathbf{x_k} \sim \sum_{j=L_\mathbf{k}}^{\infty} \frac{\mathbf{c}_j j!}{z^{j+1}}, \quad \mathbf{c}_{L_\mathbf{k}} \neq 0. \tag{3.1}$$

Remark 3.2. Since $\Re(\mu_i) > 0$ we have $\Re(\boldsymbol{\mu} \cdot \mathbf{k}) \to \infty$ as $\mathbf{k} \to \infty$. Therefore for any K, the sets of the form

$$\{\mathbf{k} \in \mathbb{N}^{m_1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) < K\} \quad \text{and} \quad \{\mathbf{k} \in \mathbb{N}^{m_1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = K\} \tag{3.2}$$

are finite.

DEFINITION 3.3 We define $S = \{\mathbf{k} : \mathbf{X_k} \not\equiv 0\}$. We define inductively the finite sets T_i (cf. Remark 3.2) and the numbers M_i as follows:

$$\begin{aligned}
T_0 &= \left\{\mathbf{k} \in S : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k}\in S} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_0\right\}, \\
T_1 &= \left\{\mathbf{k} \in S \setminus T_0 : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k}\in S\setminus T_0} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_1\right\}, \\
&\vdots \\
T_j &= \left\{\mathbf{k} \in S \setminus T_0 \setminus \cdots \setminus T_{j-1} : \Re(\boldsymbol{\mu} \cdot \mathbf{k}) = \min_{\mathbf{k}\in S\setminus T_0 \setminus\cdots\setminus T_{j-1}} \Re(\boldsymbol{\mu} \cdot \mathbf{k}) =: M_j\right\}, \\
&\vdots
\end{aligned} \tag{3.3}$$

Let also

$$r_j = \max_{\mathbf{k}\in T_j} \Re(\mathbf{a} \cdot \mathbf{k}). \tag{3.4}$$

Note also that for some $\alpha > 0$, we have

$$r_j \leq \alpha M_j. \tag{3.5}$$

Applying Remark 3.2 again, we see that

$$\bigcup_{j=0}^{\infty} T_j = S. \tag{3.6}$$

LEMMA 3.4 *We have* (*see* (1.9))

$$\mathbf{x}(z) = \sum_{\mathbf{k}\in T_0} \mathbf{C}^\mathbf{k} e^{-\mathbf{k}\cdot\boldsymbol{\mu} z} z^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x_k}(z) + O(e^{-M_1 z} z^{r_1}), \quad z \to +\infty. \tag{3.7}$$

PROOF: We write

$$\mathbf{x}(z) = \sum_{\mathbf{k}\in T_0} \mathbf{C}^\mathbf{k} e^{-\mathbf{k}\cdot\boldsymbol{\mu} z} z^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x_k}(z) + \sum_{\mathbf{k}\in S\setminus T_0} \mathbf{C}^\mathbf{k} e^{-\mathbf{k}\cdot\boldsymbol{\mu} z} z^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x_k}(z). \tag{3.8}$$

The second series is uniformly and absolutely convergent for large enough $z \in \mathbb{R}^+$ since it is bounded by the subsum of a (derivative of) a multigeometric series

$$\sum_{\mathbf{k}\in S\setminus T_0} |\mathbf{A}^{\mathbf{k}}\mathbf{C}^{\mathbf{k}} z^{\mathbf{k}\cdot\mathbf{a}}| e^{-\mathbf{k}\cdot\Re(\boldsymbol{\mu})z}\,. \tag{3.9}$$

Since (3.9) is absolutely convergent, it can be convergently rearranged as

$$\sum_{j=1}^{\infty} e^{-M_j z} \sum_{\mathbf{k}\in T_j} |\mathbf{A}^{\mathbf{k}}\mathbf{C}^{\mathbf{k}} z^{\mathbf{k}\cdot\mathbf{a}}| = \sum_{j=1}^{\infty} e^{-M_j z} z^{r_j} D_j(z) \tag{3.10}$$

(see again Definition 3.3 and Remark 3.2). It is easy to see that the $D_j(z)$ are nonincreasing in $z \in \mathbb{R}^+$, and for large enough $z > 0$ all products $z^{r_j} e^{-M_j z}$ are decreasing (cf. also (3.5)). Therefore the convergent series

$$\sum_{j=1}^{\infty} e^{-(M_j - M_1)z} z^{r_j - r_1} D_j(z) \tag{3.11}$$

is decreasing in $z > 0$ and so

$$\sum_{j=1}^{\infty} e^{-M_j z} z^{r_j} D_j(z) \le \operatorname{const} e^{-M_1 z} z^{r_1}\,.$$

□

Note. A similar strategy could also be used to show the classical Weierstrass preparation theorem.

Assume first, to get a contradiction, that we have $\mathbf{x}_0 \not\equiv 0$, and so $\mathbf{X}_0 \not\equiv 0$, so for small p we have $\mathbf{X}_{\mathbf{k}} = \sum_{j=m_0}^{\infty} \mathbf{c}_j p^j$ with $\mathbf{c}_{m_0} \neq 0$. Then, since

$$\mathbf{x}(n) = \mathbf{x}_0(n) + O(e^{-M_1 n} n^{r_1})$$

and by Remark 3.1

$$\lim_{n\to\infty} n^{-m_0-1}\mathbf{x}_0 = (m_0+1)!\, c_{m_0} \neq 0\,, \tag{3.12}$$

which contradicts $\mathbf{x}(n) = 0$ for $n \in \mathbb{N}$.

Now let

$$R_0 = \max\{\Re(\mathbf{k}\cdot\mathbf{a} - L_{\mathbf{k}} - 1) : \mathbf{k} \in T_0\} \tag{3.13}$$

and

$$T_0' = \{\mathbf{k} \in T_0 : \Re(\mathbf{k}\cdot\mathbf{a} - L_{\mathbf{k}} - 1) = R_0\}\,. \tag{3.14}$$

LEMMA 3.5 *We have*

$$\mathbf{x}(z) = \sum_{\mathbf{k}\in T_0'} \mathbf{C}^{\mathbf{k}} c_{L_{\mathbf{k}}} L_{\mathbf{k}}!\, z^{\mathbf{k}\cdot\mathbf{a} - L_{\mathbf{k}} - 1} e^{-\mathbf{k}\cdot\boldsymbol{\mu} z} + o\left(z^{R_0} e^{-M_0 z}\right) \quad \textit{for } (z \to +\infty)\,. \tag{3.15}$$

PROOF: This is an immediate consequence of Remark 3.1, Lemma 3.4, and (3.13) and (3.14). □

Completion of the Proof of Theorem 1.1

The proof now follows, by *reductio ad impossibile*, from (3.15), the assumption that $\mathbf{x}(n) = 0$ for all large enough $n \in \mathbb{N}$, the fact that by construction all $c_{L_\mathbf{k}}$ are nonzero, and the following lemma:

LEMMA 3.6 *Let* $d_\mathbf{k} \in \mathbb{C}$. *Then*

$$\sum_{\mathbf{k}\in T_0'} d_\mathbf{k} n^{\mathbf{k}\cdot\mathbf{a}-M_\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} = o(n^{R_0} e^{-K_1 n}) \quad \textit{as } n \to \infty,\ n \in \mathbb{N},$$

if and only if all $d_\mathbf{k}$ *are zero.*

PROOF: We now take $n_0 = \operatorname{card}(T_0')$, n large enough, and note that $(n+j)^b = n^b(1 + o(n^{-1}))$ if $j \le n_0$. Then a simple estimate shows that to prove the lemma, it suffices to show that the following equation cannot hold for all $0 \le l \le n_0 - 1$:

$$\sum_{\mathbf{k}\in T_0'} d_\mathbf{k} e^{-(n+l)\mathbf{k}\cdot\boldsymbol{\mu}} = q_l \tag{3.16}$$

where

$$q_l = o(e^{-nM_0}) \quad \text{as } n \to \infty,\ n \in \mathbb{N}. \tag{3.17}$$

If $n_0 = 1$, this is immediate. Otherwise, we may think of (3.16) for $0 \le l \le n_0 - 1$ as a system of equations for the $d_\mathbf{k}$ with $\mathbf{k} \in T_0'$. The determinant Δ of the system is a number of absolute value e^{-nlM_0} times the Vandermonde determinant of the quantities $\{e^{-\mathbf{k}\cdot\boldsymbol{\mu}}\}_{\mathbf{k}\in T_0'}$. In particular, for some $C > 0$ independent of n, we have that $e^{-nlM_0}|\Delta|$ is independent of n,

$$e^{-nlM_0}|\Delta| = C\Big|\prod_{\mathbf{k}_1\neq\mathbf{k}_2\in T_0'} (e^{-(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\mu}} - 1)\Big| \tag{3.18}$$

and nonzero by (1.4). Similarly, the minor $\Delta_\mathbf{k}$ of any $d_\mathbf{k}$ is bounded by $D_\mathbf{k} e^{-n(l-1)M_0}$ with $D_\mathbf{k}$ independent of n. We get $d_\mathbf{k} = o(1)$ for large n for all $\mathbf{k} \in T_0'$, and so $d_\mathbf{k} = 0$. □

3.2 Remarks on First-Order Equations

It turns out [26] that for *first-order autonomous* equations near an attracting fixed point, the series $\tilde{x}_k$ of (1.5) are mere constants and the transseries (1.5) are *classically convergent* for large enough n to actual solutions of the equation. This is a consequence of the Poincaré equivalence theorem; see [26].

Note 3.7. If $|a| = 1$, factorially divergent series do occur. In Section 3.6 we show how to use Borel summation instead of usual convergence when $a = 1$.

Assume for now that in (1.11) $G \in \mathcal{M}$ is analytic at zero, $F(0) = F'(0) = 0$ and $0 < |a| < 1$. As we mentioned, there is a one-parameter family of solutions presented as simple transseries of the form

$$x_n = x_n(C) = \sum_{k=1}^{\infty} e^{nk \ln a} C^k D_k \tag{3.19}$$

with D_k independent of C, which converge for large n. By definition their continuation to complex n is

$$x(z) = x(z; C) = \sum_{k=1}^{\infty} e^{zk \ln a} C^k D_k \,, \tag{3.20}$$

which is analytic for large enough z. To test for the Painlevé property, we proceed to find the properties of $x(z)$ for those values of z where (3.20) is no longer convergent and then find the singular points of $x(z)$.

Note. In general, although (3.20) represents a continuous one-parameter family of solutions, there may be more solutions. We also examine this issue.

Relation to Properties of the Conjugation Map

We can alternatively, and it turns out equivalently, define a continuation as follows: By the Poincaré theorem [2, p. 99] there exists a unique map φ with the properties

$$\varphi(0) = 0\,, \quad \varphi'(0) = 1\,, \quad \text{and} \quad \varphi \text{ analytic at } 0\,, \tag{3.21}$$

and such that

$$\varphi(az) = G(\varphi(z)) = a\varphi(z) + F(\varphi(z))\,. \tag{3.22}$$

The map φ is a *conjugation map* between (1.11) and its linearization

$$X_{n+1} = aX_n\,, \tag{3.23}$$

since, in view of (3.22),

$$x_n = \varphi(Ca^n) \tag{3.24}$$

for given C and n large enough, x_n is a solution of the recurrence (1.11).

We obtain a continuation of x from $\mathbb{N}$ to $\mathbb{C}$ through

$$x(z) = \varphi(Ca^z) \tag{3.25}$$

LEMMA 3.8

(i) *For equations of type* (1.11)*, continuations* (3.20) *and* (3.25) *agree.*

(ii) $x(z; C)$ *defined by* (3.20) *has only isolated movable singularities if and only if* φ *has only isolated singularities in* $\mathbb{C}$.

PROOF: Indeed, φ is analytic at the origin, and a power series expansion for large n of $\varphi(Ca^n)$ leads to a solution of the form (3.19), which obviously solves (1.11). If n_0 is large enough, it is clear that (3.19) can be inverted for C in terms of x_{n_0}, and we can also find C' so that $x_{n_0} = \varphi(C'a^{n_0})$. On the other hand, x_{n_0} uniquely determines all x_n with $n > n_0$. For equations of type (1.11), writing $x(z) = \varphi(Ca^z)$ is thus tantamount to making the substitution $n = z$ in (3.19). Note that a^z is entire and φ is analytic at zero, and the presence of a singularity of φ that is not isolated is equivalent to the presence of a similar but *movable* singularity of $x(z) = \varphi(Ca^z)$ since its position depends on C. □

Conserved Quantities

The connection between C and the equivalence map is seen as follows: Near an attracting fixed point, say 0, we have a continuous one-parameter family of solutions of (1.11) in the form (3.24).

On the other hand, the conjugation map φ is invertible for small argument by (3.21). We may then write

$$C = C(n, x_n) = \varphi^{-1}(x_n)a^{-n} =: Q(x_n)a^{-n} \tag{3.26}$$

where we see that $C(n, x_n)$ is a conserved quantity of (1.11), and $Q = \varphi^{-1}$ is analytic near zero. Clearly any equation near a stable fixed point is, in the sense of (3.26), *locally* solvable. Definition 1.2, however, requires global properties.

Note first that, from the properties of φ (or from the constancy of C), Q satisfies the functional equation

$$Q(z) = a^{-1}Q(G(z)) \,. \tag{3.27}$$

3.3 End of the Proof of Lemma 1.4(i)

PROOF: For $i = 1, 2$, let $G_i \in M$ be analytic in $\mathbb{C} \setminus E_i$, and let $\mathbb{C} \setminus E$ be the set of analyticity of $G_1 \circ G_2$. Then $E \subset \tilde{E} := E_2 \cup G_2^{-1}(E_1)$ is closed since the set of analyticity of any analytic function is open. It remains to show E is countable. Since G_2 is not identically constant, for $x \notin E_2$ there is a least $k = k(x)$ such that $G_2^{(k)}(x) \neq 0$ and then G_2 has multiplicity exactly k in a small disk D_x around x. Then $G_2^{-1}(E_1) \cap D_x$ is countable. Since for every x there is an open set D_x such that $\tilde{E} \cap D_x$ is countable, it follows that $\tilde{E}$, thus E, is also countable. In the same way, for any $a \notin E_i$ we have that $G_i^{-1}(a)$ is countable. For division, note that $1/G$ is defined wherever G is defined and nonzero. Since G is not a constant, the same argument as above shows that $G^{-1}(0)$ is countable. □

3.4 Proof of Theorem 1.5

In the following we will write $D_r(z_0)$ for the disk $\{z \in \mathbb{C} : |z - z_0| < r\}$, and D_r will denote $D_r(0)$, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

A number of notations, definitions, and results in iterations of rational maps used in the proof are reviewed in the appendix.

PROPOSITION 3.9 *Let R be a rational function of degree $d \geq 2$. Then R has infinitely many distinct periodic points.*

PROOF: By definition, points of different period are distinct, and by Lemma A.6 there are periodic points for every $n \geq 4$. □

The "If" Part of Theorem 1.5

In this direction the proof is trivial. Indeed, if G is linear fractional, then the general, nonidentically zero solution of equation (1.11) can be obtained by substituting $x = \frac{1}{y}$ in (1.11), which then becomes linear. We get

$$x_n = \left(Ca^{-n} + \frac{b}{a-1}\right)^{-1}$$

with the continuation $x(z) = (Ce^{-z\ln a} + (a-1)^{-1}b)^{-1}$, a meromorphic function.

The "Only If" Part of Theorem 1.5

Let $f = \varphi$ with φ as in Lemma 3.8(ii). For the proof we will show that if f has only isolated singularities and $f(az) = G(f(z))$, then f itself is linear-fractional. Then G is also linear-fractional since $G(w) = f(af^{-1}(w))$.

LEMMA 3.10 *If f has only isolated singularities and f is not linear-fractional, then for any large enough w, the equation $f(z) = w$ has at least two distinct roots.*

PROOF: If f is rational, then the property is immediate. Then assume that f is not rational; thus f has at least one essential singularity, possibly at infinity [18]. If f has an essential singularity in $\mathbb{C}$, then it is isolated by hypothesis and then the property follows from Theorem A.7. Then assume that f has no essential singularity in $\mathbb{C}$; thus infinity is the only essential singularity of f. If it is isolated, then Theorem A.7 applies again. Otherwise f has infinitely many poles accumulating at infinity. Since f maps a neighborhood of every pole into a full neighborhood of infinity, any sufficiently large value of f has multiplicity larger than 1. □

END OF THE PROOF OF THEOREM 1.5: Now let $G^{\circ m}$ be defined on $\mathbb{C} \setminus E_m$ and let $E = \bigcup_{m=1}^{\infty} E_m$; then E is countable and $G^{\circ m}$ is defined on $\mathbb{C} \setminus E$ for any m.

Assume f is not linear-fractional and has only isolated singularities. We let z_1 and z_2 be in $\mathbb{C} \setminus E$ and such that $f(z_1) = f(z_2)$; cf. Lemma 3.10. Then $f(az_1) = G(f(z_1)) = G(f(z_2)) = f(az_2)$ and in general $f(a^n z_1) = f(a^n z_2)$. But since $a^n z_1 \to 0$, this contradicts (3.21). □

3.5 Proof of Theorem 1.8

PROOF: The fact that $\mathcal{K}_p$ is bounded for a nonlinear polynomial map follows from the fact that after the substitution $x = \frac{1}{y}$, the map $y_{n+1} = 1/G(1/y_n)$ is attracting at $y = 0$. Thus, $\mathcal{K}_p$ is simply connected; cf. [4, theorem 5.2.3, p. 83]. Let $a_1 \in (|a|, 1)$ and *let D_ϵ be a disk such that $|G(z)| < a_1|z|$ for $z \in D_\epsilon$ and Q is analytic in D_ϵ.*

By definition, for every $z_0 \in \mathcal{K}_p$ there exists $m(z_0)$ such that $G^{[m(z_0)]}(z_0) \in D_\epsilon$. Since $G^{[m(z_0)]}(z)$ is continuous in z, there is a disk $D_{\epsilon(z_0)}(z_0)$ such that $G^{[m(z_0)]}(D_{\epsilon(z_0)}(z_0)) \subset D_\epsilon$. It follows in particular that $\mathcal{K}_p$ is open.

Since $\mathcal{K}_p$ is open and connected, it is arcwise connected. Let z_0 be arbitrary in $\mathcal{K}_p$ and let C be an arc connecting z_0 to $z = 0$. Since C is compact and

$$C \subset \bigcup_{z \in C} D_{\epsilon(z)}(z)\,,$$

there is a finite subcovering

$$C \subset \mathcal{O}_C = \bigcup_{i=1}^{N} D_{\epsilon(z_i)}(z_i)$$

with $z_i \in C$. Let M be the largest of the $m(z_i)$, $i = 1, \dots, N$. Then, by construction,

$$G^{[M]}(\mathcal{O}_C) \in D_\epsilon\,. \tag{3.28}$$

We see from (3.27) that $aQ(z) = Q(G(z)) = a^{-1}Q(G(G(z)))$, and in general, for $n \in \mathbb{N}$,

$$Q(z) = a^{-n}Q(G^{[n]}(z))\,. \tag{3.29}$$

We define $Q(z)$ in $\mathcal{O}_C$ by $Q(z) = a^{-M}Q(G^{[M]}(z))$. By (3.28), and because (3.29) holds in $\mathcal{D}_\epsilon$, this unambiguously defines an analytic continuation of Q from D_ϵ to $D_\epsilon \cup \mathcal{O}_C$. Since $\mathcal{K}_p$ is open and simply connected and since Q is analytic near zero and can be continued analytically along any arc in $\mathcal{K}_p$, standard complex analytic results show that Q is (single valued and) analytic in $\mathcal{K}_p$.

For the last part, note that the boundary of $\mathcal{K}_p$ lies in the Julia set J, which is the closure of repelling periodic points (see the appendix, Lemma A.3). Assume that x_0 is a repelling periodic point of G of period n and that x_0 is a point of analyticity of Q. Relation (3.29) implies that $Q(x_0) = 0$ and that $Q'(x_0) = a^{-n}(G^{[n]})'(x_0)Q'(x_0)$, but since $|a| < 1$ and $|(G^{[n]})'(x_0)| > 1$, this implies $Q'(x_0) = 0$. Inductively, in the same way we see that $Q^{(m)}(x_0) = 0$ for all m, which under the assumption of analyticity entails $Q \equiv 0$, which contradicts (3.21). □

3.6 Borel Summability of Formal Invariant for Logistic Map When $a = 1$

We now consider an example that cannot be reduced to the previous types, namely, when $a = 1$ and therefore the Poincaré equivalence theorem fails. In the recurrence

$$x_{n+1} = x_n(1 - x_n)\,, \tag{3.30}$$

zero is a fixed point, and it can be shown in a rather straightforward way that there are no attracting fixed points of this map or any of its iterates. However, failure of the Painlevé property can be checked straightforwardly, and Borel summability makes it possible to analyze the properties of this equation rigorously.

A formal analysis of the Painlevé property is relatively straightforward using methods similar to those in [11]. We concentrate here on properties of the conserved quantities. The recurrence $a_{n+1} = a_n(1 + a_n)^{-1}$ is exactly solvable and differs from the logistic map by $O(a_n^3)$ for small a_n. The exact solution is $n - a_n^{-1} = \text{const}$, which suggests looking in the logistic map case for a constant of the iteration in the form of an expansion starting with $C = n - a_n^{-1}$. This yields

$$C(n; v) \sim n - v^{-1} - \ln v - \frac{1}{2}v - \frac{1}{3}v^2 - \frac{13}{36}v^3 - \frac{113}{240}v^4 + \cdots\,, \tag{3.31}$$

which is indeed a formal invariant, but the associated series is factorially divergent as will appear clear shortly. Nevertheless, we can show that the expansion is Borel-summable to an actual conserved quantity in a sectorial neighborhood of $v = 0$.

THEOREM 3.11 *There is a conserved quantity C defined near the origin in $\mathbb{C} \setminus \mathbb{R}^-$ of the form*

$$C(n; v) = n - v^{-1} - \ln(v) - R(v)$$

where $R(v)$ has a Borel-summable series at the origin in any direction in the open right half-plane. $R(v)$ has a singularity barrier touching the origin tangentially along $\mathbb{R}^-$. This singularity barrier is exactly the boundary of the Leau domain of equation (3.30).

We let

$$C(n; v) := n - v^{-1} - \ln v - R(v) \tag{3.32}$$

and impose the condition that C is constant along trajectories. This yields

$$R(v) = R(v - v^2) + \frac{v}{1 - v} + \ln(1 - v) \tag{3.33}$$

where the right-hand side of (3.33) is $R(v - v^2) + O(v^2)$. The substitution

$$R(v) = h(v^{-1} - 2) \tag{3.34}$$

followed by $v = \frac{1}{x+1}$ yields

$$h(x - 1) = h(x + x^{-1}) + \frac{1}{x} + \ln\left(\frac{x}{x + 1}\right), \tag{3.35}$$

which by formal expansion in powers of x^{-1} becomes

$$h(x-1) = \sum_{k=0}^{\infty} \frac{h^{(k)}(x)}{k!} x^{-k} + \frac{1}{x} + \ln\left(\frac{x}{x+1}\right). \tag{3.36}$$

Proof of Theorem 3.11

PROPOSITION 3.12

(i) *$R(v) = h(v^{-1} - 2)$ has a Borel-summable series at the origin along $\mathbb{R}^+$. More precisely, $h(x)$ can be written in the form*

$$h(x) = \int_0^\infty e^{-px} H(p) dp \tag{3.37}$$

and where $H(p)$ is analytic at zero and in the open right half-plane $\mathbb{H} = \Re(p) > 0$ and has at most exponential growth along any ray towards infinity in $\mathbb{H}$.

(ii) *h is analytic in a region of the form $\{x : \arg(x) \neq \pi : |x| \geq \nu(\arg(x))\}$. The function ν is continuous in $(-\pi, \pi)$. (The expression of $\nu : (-\pi, \pi) \mapsto \mathbb{R}^+$ will follow from the proofs below.)*

(iii) *By* (3.37) *and Watson's lemma* [5], *h has an asymptotic power series for large x, $h(x) \sim \sum_{k=0}^{\infty} H^{(k)}(0) x^{-k}$, which is a formal solution of* (3.36).

(iv) *The function $R(v)$ is analytic in a region near the origin, the origin excluded, of the form $\mathcal{V} = \{v : \arg(v) \neq \pi, 0 < |v| < \nu^{-1}(\arg(\varphi)\}$. By* (iii) *the relation* (3.31) *is an asymptotic expansion for small $v \in \mathcal{V}$, and from* (3.37) *the power series contained there is Borel-summable.*

(v) *The function R given by* (3.34) *satisfies* (3.33).

(vi) *The function R is analytic in L_f, the Leau domain of f, and has a singularity barrier on the Julia set of f.*

PROOF: The formal inverse Laplace transform of (3.36) is the equation

$$(e^p - 1)H = \frac{1 - e^{-p} - p}{p} + \sum_{k=1}^{\infty} \frac{(-p)^k}{k!} H * 1^{*k} \tag{3.38}$$

where * denotes the Laplace-type convolution

$$F * G = \int_0^p F(s) G(p-s) ds$$

and F^{*k} is the convolution of F with itself k times. We rewrite (3.38) in the form

$$H = \frac{1 - e^{-p} - p}{p(e^p - 1)} + \frac{1}{(e^p - 1)} \sum_{k=1}^{\infty} \frac{(-p)^k}{k!} H * 1^{*k} = H_0 + \mathfrak{A}H \tag{3.39}$$

where $\mathfrak{A}$ is a linear operator. We now show that this equation is contractive in an appropriate space of functions. Let $\nu > 0$ and let $\mathcal{A}$ be the space of functions F

analytic in a neighborhood $\mathcal{N}$ of $[0, \infty)$ in the complex plane, with $F(0) = 0$, in the norm $\|F\|_\nu := \sup_{\mathcal{N}} |e^{-\nu|p|}F(p)|$. We choose $a \in (0, 2\pi)$, ϵ small, and

$$\mathcal{N} = \{p : |p| \le \epsilon\} \cup \{p : \arg(p) \in (-\tfrac{\pi}{2} + \epsilon, \tfrac{\pi}{2} - \epsilon)\}\,. \tag{3.40}$$

Since the norm $\|\cdot\|_\nu$ restricted to compact sets is equivalent to the usual sup norm, it is easy to check that $\mathcal{A}$ is a Banach space.

PROPOSITION 3.13 *For large enough ν, equation* (3.39) *is contractive in $\mathcal{A}$ in the norm $\|\cdot\|_\nu$.*

First, it is easy to see that $H_0 \in \mathcal{A}$. If $f \in \mathcal{A}$, then

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-p)^k}{k!} f * 1^{*k} &= \sum_{k=1}^{\infty} \frac{(-p)^k}{k!} \int_0^p f(s) \frac{(p-s)^{k-1}}{(k-1)!}\, ds \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(pt)(1-t)^{k-1}\, dt \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(p(1-t))t^{k-1}\, dt\,.\end{aligned} \tag{3.41}$$

It is immediate that if p is in a compact set $\mathcal{K}$ and f is analytic in $\mathcal{K}$, then the sum in (3.41) is uniformly convergent in $\mathcal{K}$ and analytic in p. Furthermore, the sum is $O(p^3)$ for small p since $f \in \mathcal{A}$. Now we see that

$$\begin{aligned}&\left| e^{-\nu|p|} \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 f(p(1-t))t^{k-1}\, dt \right| \\ &\quad = \left| \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{k!(k-1)!} \int_0^1 e^{-\nu|p|(1-t)} f(p(1-t))t^{k-1} e^{-\nu|p|t}\, dt \right| \\ &\quad \le \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^{2k}}{k!(k-1)!} \int_0^1 t^{k-1} e^{-\nu|p|t}\, dt \\ &\quad \le \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^{2k}}{k!(k-1)!} \int_0^\infty t^{k-1} e^{-\nu|p|t}\, dt \\ &\quad = \|f\|_\nu \sum_{k=1}^{\infty} \frac{|p|^k}{k!\nu^k} \\ &\quad \le \|f\|_\nu \frac{|p|}{\nu} e^{\frac{|p|}{\nu}}\end{aligned} \tag{3.42}$$

and thus

$$\|\mathfrak{A}\| \le \text{const}\, \nu^{-1} \tag{3.43}$$

for sufficiently large ν, where we took into account the exponential decrease of $(e^p - 1)^{-1}$ for large p in $\mathcal{N}$. Thus the equation has a unique fixed point $H \in \mathcal{A}$. In particular, the Laplace transform $h(x) = \mathcal{L}H = \int_0^\infty e^{-xp}H(p)dp$ is well-defined and analytic in the half-plane $\Re(x) > \nu$. It is now immediate to check that $h(x)$ satisfies equation (3.36). □

4 Julia Sets for the Map (1.13) for $a \in (0, 1)$

It is convenient to analyze the superattracting fixed point at infinity; the substitution $x = \frac{1}{y}$ transforms (1.13) into

$$y_{n+1} = -\frac{y_n^2}{a(1 - y_n)}\,. \tag{4.1}$$

For small y_0, the leading-order form of equation (4.1) is $y_{n+1} = -a^{-1}y_n^2$ whose solution is $-y_0^{2^n}a^{-2^{n-1}-1}$. It is then convenient to seek solutions of (4.1) in the form $y_n = -G(y_0^{2^n}a^{-2^n})$ whence the initial condition implies $G(0) = 0$, $G'(0) = a$. Denoting $y_0^{2^n}a^{-2^n} = z$, the functional relation satisfied by G is

$$G(z^2) = \frac{G(z)^2}{a(G(z) + 1)}\,, \quad G(0) = 0, \quad G'(0) = a\,. \tag{4.2}$$

LEMMA 4.1 [12] *There exists a unique analytic function G in the neighborhood of the origin satisfying* (4.2)*. This G has only isolated singularities in $\mathbb{C}$ if and only if $a \in \{-2, 2, 4\}$. In the latter case,* (1.13) *can be solved explicitly.*

If $a \notin \{-2, 2, 4\}$, then the unit disk is a barrier of singularities of G.

LEMMA 4.2 *G is analytic in the open unit disk S_1 and Lipschitz-continuous in $\overline{S_1}$.*

PROOF: Lemma 4.1, proven in [12], guarantees the existence of some disk S_r centered at zero, of radius $r \le 1$, where G is analytic, and it is shown that inside that disk we have (cf. also 4.2)

$$G(z) = U(G(z^2))\,, \quad 2U(s) := s + (a^2s^2 + 4s)^{\frac{1}{2}} \tag{4.3}$$

(with the choice of branch consistent with $G(0) = 0, G'(0) = a$). If $r < 1$, then (4.3) provides analytic continuation in a disk of radius $r^{1/2} > r$ if $a^2G(z)^2 + 4aG(z) \ne 0$ in S_r.

Remark 4.3. $G(z_0) = 0$ in S_1 if and only if $z_0 = 0$.

Indeed, assume $0 \ne z_0 \in S_r$ and $G(z_0) = 0$. Then we find from (4.2) that $G(z_0^{2^n}) = 0$, which is impossible since G is analytic at zero and $G'(0) = a$.

We are left to examine the possibility $G(z_0) = -4a^{-1}$ with $z_0 \in S_r$.

Remark 4.4. $R(x) = x^2/(a(x + 1))$ is well-defined and increasing on the interval $(-\infty, -4a^{-1})$.

The assumption $G(z_0) = -4a^{-1}$ thus implies that the values $G(z_0^{2^n})$ are in $\mathbb{R}^-$ and decrease in n, which is again impossible because G is analytic at zero and $G(0) = 0$.

We now show that G is bounded in S_1. Indeed, by (4.3) we have

$$|G(z)| \le U(|G(z^2)|)\,; \tag{4.4}$$

on the other hand, a calculation shows that

$$U(s) \le \frac{a}{1-a} \quad \text{for } s \in \left[0, \tfrac{a}{1-a}\right]. \tag{4.5}$$

Since $G(0) = 0$ and G is analytic in S_1, (4.4) and (4.5) imply that

$$\sup_{z\in S_1} |G(z)| \le \frac{a}{1-a}\,. \tag{4.6}$$

We next prove that G is injective. As a first step we have the following:

Remark 4.5. $G' \neq 0$ in S_1.

Indeed, otherwise differentiating (4.2) shows there would exist a sequence $z_n \to 0$ such that $G'(z_n) = 0$.

Now, G is injective in a neighborhood of the origin since $G'(0) = a$. Let $z_1 \in S_1$ be a point of smallest modulus such that there exists $z_2 \neq z_1 \in S_1$ with $G(z_1) = G(z_2)$. For z_1 to exist, we need, again by (4.2), that $z_1^2 = z_2^2$ and thus $z_1 = -z_2$. Since $G' \neq 0$, by the open mapping theorem, the image under G of arbitrarily small disks around z_1 and $-z_1$ overlap nontrivially. For some C and any ϵ, there exist therefore infinitely many z_i with $|z_1 - z_i| < \epsilon$ such that $G(z_i) = G(z_i')$ and $|z_i' - (-z_1)| < C\epsilon$. The same argument using (4.2) shows that $z_i' = -z_i$. But since $G(z) = G(-z)$ for infinitely many $z \in S_1$ accumulating at z_1, G would be even, which is not the case since $G'(0) = a$. We now need two lower bounds.

PROPOSITION 4.6 *For* $a \in (0, \frac{1}{2})$

$$(1-|z|)^{1-\log_2(2-a)}G'(z)$$

is bounded in S_1.

PROOF: The function $H = \frac{1}{G}$, which, by Remark 4.3, is analytic in $S_1 \setminus 0$, satisfies

$$H(z^2) = aH(z)(1+H(z))\,. \tag{4.7}$$

Let

$$m_n = \max\left\{|H(z)| : |z| \in [2^{-\frac{1}{2^n}}, 2^{-\frac{1}{2^{n+1}}}]\right\}.$$

Equation (4.7) gives

$$m_{n+1} \le \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{m_n}{a}}\,,$$

and it is easy to see that this implies

$$\limsup_{n\to\infty} m_n \le 1 + a^{-1}\,. \tag{4.8}$$

We have

$$G'(z) = 2az\,\frac{G'(z^2)(1+G(z))^2}{G(z)(2+G(z))} \tag{4.9}$$

so that

$$|G'(z)| \le |G'(z^2)| \max_{(1-a)|y|\le a} \left|\frac{2a(1+y)^2}{y(2+y)}\right| = \frac{2}{2-a}|G'(z^2)|$$

if $a \le \frac{1}{2}$ from which Proposition 4.6 follows immediately. □

Remark 4.7. A straightforward way to extend the result for larger values of $a < 1$ is to replace (4.9) by a corresponding equality obtained from a higher-order iterate of (4.7).

LEMMA 4.8 *G gives a conformal transformation of S_1 onto a bounded region $\mathcal{K}_p$ whose boundary $\partial\mathcal{K}_p$ is a Lipschitz-continuous, nowhere differentiable curve.*

5 Behavior at the Singularity Barrier

PROPOSITION 5.1 *There is $\delta > 0$, a real analytic function Ψ, periodic of period $\ln 2$, and an analytic function Φ, $\Phi'(0) = 1$, such that for $|\arg(1-z)| < \delta$ equality* (1.15) *holds.*

PROOF: Let $\omega = 2\pi/\ln 2$, $\beta = \log_2(2-a)$. With $z_0 \in (0,1)$ and $z_n = z_0^{1/2^n}$, the sequence $G_n = G(z_n)$ is increasing and bounded by L; see (4.3). It follows immediately from (4.3) and (4.4) that

$$L - G_n := \delta_n \downarrow 0 \quad \text{as } n \to \infty \quad \left(L := \frac{a}{1-a}\right). \tag{5.1}$$

From (4.8) we have, with $C_1 = (1-a)^3/[a(2-a)] + C_2$, $C_2 = 1-a$,

$$\delta_{n+1} = \frac{1}{2-a}\left[\frac{1 - C_2\delta_{n+1}}{1 - C_1\delta_{n+1}}\right]\delta_n\,. \tag{5.2}$$

Equations (5.1) and (5.2) imply that for any $\epsilon > 0$ we have

$$\delta_n = o((2-a-\epsilon)^{-n}) \quad \text{as } n \to \infty\,. \tag{5.3}$$

Let

$$\delta_n = \ln^\beta\left(\frac{1}{z_n}\right)e^{\theta_n} = 2^{-n\beta}\ln^\beta\left(\frac{1}{z_0}\right)e^{\theta_n}\,; \tag{5.4}$$

cf. (5.1). Now

$$|e^{\theta_n - \theta_{n+1}} - 1| = \frac{(C_1 - C_2)\delta_{n+1}}{1 - C_2\delta_{n+1}} = O(\delta_n) \quad \text{as } n \to \infty\,,$$

and by (5.3), θ_n is convergent, $\theta_n \to \Theta$. Since $\theta_{n+1} - \theta_n \to 0$, it follows that

$$\Theta(z_0^2) = \Theta(z_0)\,. \tag{5.5}$$

Analyticity

We let $1 - z_1$ be sufficiently small so that

$$\delta_n \leq c\alpha^n \tag{5.6}$$

with $\alpha < 1$ and c small enough so that the term in square brackets is sufficiently close to 1 for all $n \geq 0$ and $|z_0 - z_1| \leq \epsilon_1$ (cf. (5.3)); this amounts to a shift in n. If ϵ_1 is small enough, then it is easy to check that equation (5.2) is a contractive mapping in the ball of radius c $S_{\epsilon_1} = \{\zeta : |\zeta| \leq \epsilon_1\}$ in Banach space $l_{\infty,\alpha}(\mathbb{N})$ of vectors $\mathbf{v}(n;\zeta)$ analytic in $\zeta = z_0 - z_1$ with respect to the norm

$$\|\mathbf{v}\| = \sup_{\substack{n\geq 1\\ |\zeta|\leq\epsilon_1}} |\mathbf{v}(\mathbf{n},\zeta)\alpha^{-\mathbf{n}}|$$

and local analyticity in a neighborhood of the interval $[z_0, \sqrt{z_0}]$. By periodicity, real analyticity follows immediately and relation (5.5) is preserved. □

End of Proof of Theorem 1.10(iii)

We use the information obtained in Section 5. Let $e^{\theta_n} = (1 + w_n)e^{\Theta}$; given $\delta > 0$ we choose n_0 large enough and ϵ_2 so that $|w_n(z_0)| < \delta$ if $|z - z_0| < \epsilon_2$ and $n \geq n_0$. We let $h = e^{2\Theta}$, $\varepsilon_n = 2^{n\beta}$, $s = \ln^{\beta}(1/z_0)$, $c = C_1 - C_2$, and $C = c - C_2$ and obtain

$$w_n = \frac{Ce^{2\Theta}s\varepsilon_n}{1 - \varepsilon_n C_2 e^{2\Theta}s} + w_{n+1}\,\frac{1 + 2Ce^{2\Theta}s\varepsilon_n - CC_2e^{4\Theta}s^2\varepsilon_n^2 + w_{n+1}Cse^{2\Theta}\varepsilon_n(1 - \varepsilon_nC_2se^{2\Theta})}{1 - 2\varepsilon_nse^{2\Theta} + 2\varepsilon_n^2s^2e^{4\Theta} - w_{n+1}C_2se^{2\Theta}(1 - \varepsilon_nC_2se^{2\Theta})}\,. \tag{5.7}$$

As in Section 5, a contractive mapping argument shows that $\mathbf{w} = (w_n, w_{n+1}, \ldots)$ is analytic in $se^{2\Theta}$, if s is small enough. The conclusion now follows from the definition

$$G(z_0^{2^{-n_0}}) = L + s\delta_{n_0}$$

and (5.4), (5.5), Section 5, and the substitution $e^{2\Theta(\cdot)} = \Psi(\ln(\ln(\cdot))$. Formula (1.17) follows immediately from (1.15). □

Remark 5.2. With $z_n = z_0^{1/2^n}$, $\tau_n = \tau(z_n)$ (cf. (1.15), and $g_n = G(z_n) - L$, we have

$$\Psi(\ln\ln z_0) = \lim_{N\to\infty} \frac{\dfrac{g_{N+1}}{\tau_{N+1}} - \dfrac{g_N}{\tau_N}}{\tau_{N+1} - \tau_N}\,. \tag{5.8}$$

Appendix: Iterations of Rational Maps

We introduce a number of definitions and results for iterations of rational maps, which are treated in much more detail and generality in [4, 32]. We shall illustrate the main concepts on the simple case $G = ax(1-x)$. In Figure 1.1, the interior (in the complex plane) of the fractal curves is a set invariant under G and with the further property that starting with z_0 inside the m^{th} iterate of G at z_0, $G^{\circ m}(z_0)$, converges to zero as $m \to \infty$. These are *stable fixed domains* of G.

Consider the polynomial map G. A Fatou domain of G is a stable fixed domain V of G characterized by the property that $G^{\circ n}$ converges *in the chordal metric* on the Riemann sphere $\mathbb{C}_\infty$ to a fixed point of G, locally uniformly in V.

DEFINITION A.1 [4, p. 50] Let G be a nonconstant rational function. The *Fatou set* of G is the maximal open subset of $\mathbb{C}_\infty$ on which $\{G^{\circ n}\}$ is equicontinuous and the *Julia set* of G is its complement in $\mathbb{C}_\infty$.

A Fatou domain is a *Leau* domain (or a parabolic basin) if $x_0 \in \partial V$ and the multiplier of x_0 (the derivative at x_0) is $\lambda = 1$ [32, p. 54]. In Figure 1.1 this happens for $a = 1$.

The Julia set can be characterized by the following property:

LEMMA A.2 [4, p. 148] *Let G be a rational map of degree d (cf. Definition* A.4) *where $d \geq 2$. Then J is the derived set of the periodic points of G.*[2]

Under the assumptions above, we have the following:

LEMMA A.3 [4, p. 148] *J is the closure of the repelling points of G.*

DEFINITION A.4 [4, p. 30] If $R = \frac{P}{Q}$ where P and Q are polynomials, then the degree of the rational function R is $\max\{\deg(P), \deg(Q)\}$.

DEFINITION A.5 [4] If R is a rational function and $R^{\circ m}=R \circ R \circ \cdots \circ R$ n times, then a periodic point of period n of R is a point z such that $R^{\circ m} z = z$ and $R^{\circ m} z \neq z$ if $m < n$. A periodic point of R is a point of some period $n \geq 1$.

We also use the following result of I. N. Baker:

LEMMA A.6 [3, 4] *Let R be a rational function of degree $d \geq 2$, and suppose that R has no periodic points of period n. Then (d, n) is one of the pairs*

$$(2, 2), (2, 3), (3, 2), (4, 2)$$

(*moreover, each such pair does arise from some R in this way*).

[2] By definition the derived set of a set E consists exactly of the points z that are limits of sequences $\{z_n\}$ where the $z_n \in E$ are distinct.

Further Results Used in the Proofs

THEOREM A.7 (Big Theorem of Picard, Local Formulation [18, 30]) *If f has an isolated singularity at a point z_0 and if there exists some neighborhood of z_0 where f omits two values, then z_0 is a removable singularity or a pole of f.*

THEOREM A.8 (Picard-Borel [27]) *If φ is any nonconstant function meromorphic in $\mathbb{C}$, then φ avoids at most two values* (*infinity included*).

All we need in the present paper is that at most two finite values are excluded. This is immediately reduced to the more familiar Picard theorem by noting that if λ is an excluded value of f, then $1/(f-\lambda)$ is entire.

A.1 Proof of Proposition 1.7

By Theorem 1.5, (1.11) does not have the Painlevé property at some stable fixed point if and only if G is not linear-fractional, in which case (1.11) fails to have the Painlevé property at any other stable fixed point. More generally, Proposition 1.7 follows from the following result:

LEMMA A.9 *If $G^{\circ m}$ is of the form* (1.12)*, then G is of the form* (1.12).

PROOF: Since (1.12) is one-to-one, the conclusion follows from the remark that if G is not linear-fractional, then $G(z)$ has multiplicity greater than 1 for all sufficiently large z (and then the same holds for $G^{\circ m}(z)$). Indeed, assume that G is not linear-fractional. If G is rational, then the conclusion is obvious. If the set of singularities of G is finite, then they are all isolated and at least one is an essential singularity (otherwise G is rational [18]) and Theorem A.7 applies.

So we may assume the set of singularities is infinite. Since by assumption this set is closed and countable, it contains infinitely many isolated points. (Indeed, a set that is closed and dense in itself, i.e., a *perfect set*, is either empty or else uncountable.) Then if G has an isolated essential singularity, Theorem A.7 applies, and if not then there are infinitely many poles of G. In the latter situation any sufficiently large value of G has multiplicity larger than 1, since G maps a neighborhood of every pole into a full neighborhood of infinity. □

A.2 Completion of the Proof of Proposition 3.12

Part (ii) merely follows from formula (3.37) and elementary contour deformation in the integral. Parts (iii) and (iv) are straightforward.

After the transformation $v=-u+\frac{1}{2}$ the iteration associated to our map f is equivalent to that of the quadratic map $q(u)=u^2+\frac{1}{2}$.

Part (vi) follows from the following lemma:

LEMMA A.10 [32, p. 174] *The Leau domain of q is the filled-in* (*interior of the*) *Julia set $\mathcal{K}_p$ of q.*

PROOF OF PROPOSITION 3.12(v): Let $H(v) = R(v) + v^{-1} + \ln v$, defined and analytic in $\mathcal{V}$. By definition we have $H(v_{n+1}) = H(v_n) + 1$, i.e.,

$$H(v) = H(f(v)) - 1 = H(v - v^2) - 1 \tag{A.1}$$

and clearly R and H have the same type of singularities in $\mathbb{C} \setminus \mathbb{R}^- \setminus \{0\}$.

If $z_0 \in L_f$ we have by definition $|z_n| = |f^{\circ m}(z_0)| \to 0$. Then, we choose ϵ small enough and N so that $|z_n| < \epsilon$ for $n > N$. Since we must have for some $n > N$ that $|z_{n+1}| < |z_n|$, then $|1 - z_n| < 1$ and thus $\arg(z_n) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. A direct calculation shows that then $|\arg(z_{n+1})| < |\arg(z_n)|$ and thus, if $m > n$, then $\arg(z_m) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus by Proposition 3.12(ii) and (iv), eventually $z_n \in \mathcal{V}$. We know that $\mathcal{V}$ is a domain of analyticity of R. By (A.1), if H is analytic at $z_{n+1} = z_n - z_n^2$, then H is analytic at z_n and by induction H is analytic at z_0. Since L_f is simply connected, we have that H, and thus R, is analytic in L_f, as in the proof of Theorem 1.8.

On the other hand, if we assume that $v \in \partial L$ is a periodic point of f, say of period N, and that R, and thus H, is analytic there, relation (A.1) implies that H is analytic at any point on the orbit of v and furthermore $H(v) = H(v) - N$, a contradiction. Since the closure of the periodic points is ∂L, ∂L is a singularity barrier of H. Furthermore, ∂L is in the exterior of $\mathcal{V}$, and since $\partial\mathcal{V}$ touches the origin tangentially along $\mathbb{R}^-$, so does ∂L since $0 \in \partial L$. □

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