10-15-2002

1. Let f be analytic in the open unit disk $D(0,1)$ and $\gamma : [0,1] \mapsto D(0,1)$ be continuous. If $\mathcal{P}$ is a partition of the form $0 = t_1 < t_2 < ... < t_n = 1$ then its (usual) norm is $\|\mathcal{P}\| = \max_j\{t_{j+1} - t_j\}$. (Note that, by uniform continuity, $\max_j\{|\gamma(t_{j+1}) - \gamma(t_j)|\} \to 0$ as $\|\mathcal{P}\| \to 0$.) Is it true or false that, under the stated conditions,

$$\int_\gamma f = \lim_{\|\mathcal{P}\|\to 0} \sum_j f(\gamma(t_j))(\gamma(t_{j+1}) - \gamma(t_{j+1}))?$$

2* *Partial fraction decomposition.* Let P and Q be polynomials such that $n = \deg P > \deg Q$ and assume that the roots of P, $z_1, ..., z_n$, are distinct. Use analytic methods to show that there exist (unique) constants $A_1, ..., A_n$ such that

$$\frac{Q(z)}{P(z)} = \sum_{i=1}^{n} \frac{A_i}{z - z_i}, \quad z \in \mathbb{C}$$

How would you deal with roots of higher multiplicity (here you may simply outline the main steps of an argument).

3* *Writing entire functions in terms of their roots.* We will see that that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \quad (*)$$

a natural generalization of the decomposition of polynomials in terms of roots. (A similar result holds for general entire functions, which are determined by their roots up to multiplication by a suitable exponential of an entire function.)

(a) Show that for $s \in \mathbb{R}^+$ we have

$$\sum_{n=2}^{\infty} \frac{1}{s + n^2} \le \frac{\pi}{2\sqrt{s}}$$

(one way is to use an integral estimate) and then

$$\sum_{n=1}^{\infty} \ln(1 + s/n^2) \le \pi\sqrt{s} + \ln(s+1)$$

(b) Show that the sequence

$$g_N(z) = \pi z \prod_{n=1}^{N} \left(1 - \frac{z^2}{n^2}\right)$$

is uniformly convergent on compact sets in $\mathbb{C}$ to an entire function, $g(z)$.

(c) Prove that $h(z) = \sin(\pi z)/g(z)$ extends as an entire function in $\mathbb{C}$ which has no zeros and that $h(z) = e^{H(z)}$ for an entire function H. Use (a) to conclude that $(1 + |z|)^{-1} H(z)$ is bounded in $\mathbb{C}$ and thus H is a polynomial of degree at most one.

(d) From (c) and the properties of sin and g conclude that $h = 1$ in $\mathbb{C}$.

(e) Use (*) to show that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Similarly, one can show that

$$\sum_{n=1}^{\infty} n^{-4} = \pi^4/90, \quad \sum_{n=1}^{\infty} n^{-6} = \pi^6/945, \quad \text{etc.}$$

Outline a strategy.