1. First view at Cauchy's integral formula for holomorphic functions. 2. Holomorphic implies analytic.

1*. Assume f is holomorphic in the disk $\mathcal{D}(0,r)$. Let

$$h(z) = \begin{cases} [f(z) - f(0)]/z & (z \neq 0) \\ f'(0) & (z = 0) \end{cases}$$

(a) Show that h is continuous in $\mathcal{D}(0,r)$ and differentiable at all points $z \neq 0$.

(b) Let $H(z) = \int_0^z h(s)ds$ where the integral is taken along any polygonal arc with sides parallel to the axes. Show that H is correctly defined.

(c) Show that $H' = h$ throughout $\mathcal{D}(0,r)$, thus h is holomorphic in $\mathcal{D}(0,r)$.

(d) Show that if γ is a closed curve in $\mathcal{D}(0,r)$ then $\int_\gamma h = 0$. In particular, if γ is a circle around zero, then

$$f(0) = \frac{1}{2\pi i}\int_{|z|=r'} \frac{f(s)}{s} ds$$

This clearly generalizes to functions holomorphic in other disks, and thus, if f is analytic in $\mathcal{D}(z,r)$ and $r' < r$,

$$f(z) = \frac{1}{2\pi i}\int_{|s-z|=r'} \frac{f(s)}{s-z} ds$$

(e) Show that

$$\int_C \frac{ds}{s} = 2\pi i$$

for any circle containing zero. (A straightforward calculation would work, but there are simpler ways.) Show that this implies for that $|z| < r'$ the particular instance of *Cauchy's integral formula* for holomorphic functions:

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{|z|=r'} \frac{f(s)}{s-z} ds \ (\dagger)}$$

2.* We have

$$\frac{1}{s-z} = \frac{1}{s}\,\frac{1}{1-(z/s)} = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{z}{s}\right)^k$$

Show that convergence is *uniform* and thus you can integrate term in by term the series. Use (†) to show that for $|z| < r'$ and $r' < r$ we have

$$\boxed{f(z) = \sum_{k=0}^{\infty} f_k\, z^k}$$

where the series is convergent for $|z| < r$ and therefore f is *analytic* in the disk of radius r, and we have

$$\boxed{f_k = \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(s)}{s^{k+1}} ds}$$

for any $r' < r$.