HOMEWORK SET 3. PLEASE TURN IN THE * PROBLEMS (OR * PARTS).

1. Which of the following subsets of $\mathbb{C}$ are connected? Which of them are regions? Explain.

(a) $\mathbb{C} \setminus \mathbb{Z}$

(b*) $\mathbb{C} \setminus \mathbb{Q} + i\mathbb{Q}$

(c) $\{z : |z| > 1\}$

2. Assume $f(z) = \sum_{k \geq 0} c_k z^k$ where the series is convergent in the open unit disk.

(a) Show that if the n-th derivative has the property $|f^{(n)}(z)| \to \infty$ as $z \to 1^-$ then f is not analytic at $z = 1$.

(b*) Show that if there exist two positive constants M_1, M_2 such that $M_1\sqrt{1-|z|} < |f(z)| < M_2\sqrt{1-|z|}$ for $|z| < 1$, then f cannot be analytic at $z = 1$.

3. Define the function $f(z) = \sqrt{1+z}$ in the open unit disk through the usual convergent binomial series

$$f = 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1} \left(\frac{1}{2} - j\right)}{n!} z^n$$

(Note that the conclusions of Problem 2 imply that f is not analytic at $z = -1$.)

(a*) Show that f has analytic continuation in the region $\{z : z \notin (-\infty, -1)\}$.

(b) There are many half-lines (rays) through $\{-1\}$ with the same feature. Check *e.g.* that f has analytic continuation in the region $\{z : i(z+1) \notin \mathbb{R}^+\}$

(c*) Show that $f(z) = \sqrt{1+z}$ has no analytic continuation in any region of the form $\{z : |z+1| > a\}$, where $a \geq 0$. (Note by contrast that the function $g(z) = (z+1)^{-1}$ has analytic continuation in $\mathbb{C} \setminus \{0\}$.)

4*. (a) Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Show that f has no analytic continuation in any region containing a part of the unit circle. (Hint: Look at $f\left(\rho e^{2\pi i m/2^n}\right)$ (with $m, n \in \mathbb{N}$) as $\rho \to 1^-$.)

(b) Let $P_n(z) = \prod_{j=1}^{n}(1 - z^{2^n})$. Show that P_n converges uniformly to a function $f(z)$ in any disk of the form $\{z : |z| \leq r\}$ if $r < 1$. Then show that $f(z)$ is given by a convergent power series in the open unit disk. Finally, prove that f has no analytic continuation in any region containing a part of the unit circle.