Applied Stochastic Processes

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1. **Problem:** Find $E(\int_0^T a(t)X(t)\;dt)^n$, $n = 0, 1, \ldots$, where X is a Gaussian process with continuous covariance $R(s,t)$, mean zero, and a is a continuous function.

   Solution: Note that $a(t)X(t)$ is also a Gaussian process with continuous covariance $a(s)a(t)R(s,t)$ and mean zero, so without loss of generality we need only consider $M = \int_0^T X(t)\;dt$.

   To find $E(M^n)$, observe that

   $$M^n = \lim_{\Delta_\cdot \to 0} \sum_{j_\cdot = 1 \ldots m_\cdot} X(i_{j_1}) \ldots X(i_{j_n}) \cdot (i_{j_1} - i_{j_1 - 1}) \ldots (i_{j_n} - i_{j_n - 1}),$$

   where the $\{i_{j_\cdot}\}$ are partitions of $[0,T]$, and $\Delta_k = \max(i_{j_k} - i_{j_k - 1})$.

   Taking expectations,

   $$\begin{aligned} E(M^n) &= E\left(\lim_{\Delta_\cdot \to 0} \sum_{j_\cdot = 1 \ldots m_\cdot} X(i_{j_1}) \ldots X(i_{j_n}) \cdot (i_{j_1} - i_{j_1 - 1}) \ldots (i_{j_n} - i_{j_n - 1})\right) \\ &= \lim_{\Delta_\cdot \to 0} E\left(\sum_{j_\cdot = 1 \ldots m_\cdot} X(i_{j_1}) \ldots X(i_{j_n}) \cdot (i_{j_1} - i_{j_1 - 1}) \ldots (i_{j_n} - i_{j_n - 1})\right) \\ &= \lim_{\Delta_\cdot \to 0} \sum_{j_\cdot = 1 \ldots m_\cdot} R(i_{j_1}, \ldots, i_{j_n}) \cdot (i_{j_1} - i_{j_1 - 1}) \ldots (i_{j_n} - i_{j_n - 1}) \\ &= \int_{[0,T]^n} R(t_1, \ldots, t_n)\;dt_\cdot. \end{aligned}$$

   This is a Gaussian process, and so the covariance $R(t_1, \ldots, t_n)$ can be expressed in terms of the $R(t_j, t_k)$. Doing this and simplifying,

   $$\begin{aligned} E(M^2) &= \int_0^T \int_0^T R(s,t)\;ds\;dt \\ E(M^{2n+1}) &= 0 \\ E(M^{2n}) &= \frac{1}{n!}\binom{2n}{2, \ldots, 2} E(M^2)^n. \end{aligned}$$

In the languague of physics and the annihilation and creation operators

$$E(M^n) = \langle\ 0\ |(a+a^*)^n|\ 0\ \rangle \cdot E(M^2)^{n/2},$$

which is the number of 1-valent Feynman diagrams with n vertices multiplied by the volume of the covariance ellipsoid. This is a nice example of a quantum field theory calculation done in a finite-dimensional, rather than infinite-dimensional, Hilbert space.

□

2. **Problem:** Prove that $\sup_{0<t<1}(W(t)) > 0$ with probability 1 for a standard Brownian motion W by either using the optional sampling theorem or the reflection principle. Also find $\sup_\tau E(W(\tau))$, where the supremum is over all stopping times $\tau \leq 1$.

   Solution - Reflection:

   Let $a > 0$. By the reflection principle

$$\begin{aligned} p_a &= P\left(\sup_{0<t<1}\{W(t)\} \geq a\right) \\ &= 2 \cdot P\left(W(1) \geq a\right) \\ &= \frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2}\,dx. \end{aligned}$$

   It follows that

$$\begin{aligned} P\left(\sup_{0<t<1}\{W(t)\} > 0\right) &= \lim_{a\to 0^+} p_a \\ &= \lim_{a\to 0^+} \frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2}\,dx \\ &= 1. \end{aligned}$$

   □

   Solution - Optional Stopping Theorem:

   Let $\tau_a = \min\{t > 0 : W(t) = a\}$, and let $\tau = \tau_0 \wedge 1$ and $p = P(\tau < 1)$. Now we have that $E(W(\tau)) = p \cdot 0 + (1-p) \cdot e$, where

$$e = E(W(1) \mid W(t) < 0 \text{ for } 0 < t < 1) \Longrightarrow e < 0.$$

   But by the optional stopping theorem $E(W(\tau)) = E(W(0)) = 1$, and so $p = 1$.

   □

3. **Problem:** Show that the simple one-dimensional random walk $S_0 = 0, S_n = \sum_{j=0}^{n} X_j$, where the X_j are i.i.d. Bernoulli r.v.'s with $P(X_j = \pm 1)$, hits every integer $a > 0$ with probability 1 by using optional sampling and the stopping times $\tau = \min(\tau_{-N}, \tau_a)$, where for any b, τ_b is the first n such that $S_n = b$, or ∞ if there is no such n.

   Solution: Let $p = P(\tau = \tau_a)$, then $E(S_\tau) = p \cdot a + (1-p) \cdot (-N)$. By the optional sampling theorem $E(S_\tau) = E(S_0) = 0$, and so

$$p \cdot a + (1-p) \cdot (-N) = 0 \Longrightarrow p = N/(a+N).$$

   Letting $N \to \infty$, we see that $p \to 1$, and so this random walk hits every $a > 0$ with probability 1. □

4. **Problem:** Find the expectation of the random variable $\tau = \min(\tau_{-N}, \tau_a)$ in the problem above by making up a martingale that will give you the answer via the optional sampling theorem.

   Solution: Consider the martingale $Y_n = {S_n}^2 - n$ and let p be as above. Here

$$E(Y_\tau) = p \cdot (a^2 - E(\tau_a)) + (1-p) \cdot (N^2 - E(\tau_{-N})).$$

   Again, by the optional sampling theorem $E(Y_\tau) = E(Y_0) = 0$, and so $p \cdot (a^2 - E(\tau_a)) + (1-p) \cdot (N^2 - E(\tau_{-N}) = 0$, giving

$$E(\tau) = p \cdot E(\tau_a) + (1-p) \cdot E(\tau_{-N}) = p \cdot a^2 + (1-p) \cdot N^2.$$

   From the previous result $p = N/(a+N)$, and so finally

$$E(\tau) = aN(N+a)/(a+N) = aN.$$

   It follows that although this random walk hits every $a > 0$ with probability 1, the expected time to the first hit is infinite.

   As a final comment, the martingale $Z_n = {S_n}^4 - 2n \cdot {S_n}^2 - n^2 + 2n$ gives the result

$$E(\tau^2) = \frac{1}{3} aN(a^2 - aN + N^2 + 2).$$

   □