Applied Stochastic Processes

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1. **Problem:** Find the n step transition probabilities for the discrete time Markov chain on two states with transition probability matrix P, where all the 4 entries of P are strictly positive, and also find the invariant measure.

   Solution: For convenience, let the transition matrix P be written as

   $$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

   P has the characteristic polynomial $\chi(\lambda) = \lambda^2 - (2-\alpha)\lambda + (1-\alpha)$, where $\alpha = p+q$. Now $\chi(\lambda) = (\lambda - 1)(\lambda - (1-\alpha))$, and so P has the two eigenvalues $\lambda_1 = 1, \lambda_2 = 1-\alpha$ (which are distinct, as $\alpha \neq 0$) with the corresponding left eigenvectors $x_1 = \begin{pmatrix} q & p \end{pmatrix}$ and $x_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}$.
   As P has a full eigenspace P is diagonalizable and $P = QDQ^{-1}$, where

   $$Q = \begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha \end{pmatrix},\ Q^{-1} = \frac{1}{\alpha}\begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix}.$$

   It follows that the n step transition probability matrix P^n can be written as

   $$P^n = QD^nQ^{-1} = \frac{1}{\alpha}\begin{pmatrix} q + p(1-\alpha)^n & -pq + pq(1-\alpha)^n \\ -1 & p + q(1-\alpha)^n \end{pmatrix}.$$

   Finally, an invariant measure π_0 is a normalized left eigenvector for the eigenvalue 1. Here it's unique, with

   $$\pi_0 = \begin{pmatrix} q/(p+q) & p/(p+q) \end{pmatrix}.$$

   □

2. **Problem:** Let $X(t), t \geq 0$ denote a continuous time Markov chain on 3 states, $\{1, 2, 3\}$, and suppose the rate of leaving state i for state $j \neq i$ is r_{ij}, with $r_{12} = 1$, $r_{23} = 2$, $r_{31} = 3$, and otherwise $r_{ij} = 0$. Find the Laplace transform of the transition probability,

   $$\hat{P}_1(s) = \int_0^\infty P(X(t) = 1 \mid X(0) = 1) \cdot e^{-st} dt,$$

and the invariant measure.

Solution: The infinitesimal generator here is

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & 0 & -3 \end{pmatrix}.$$

Starting in state 1 at $t = 0$ the differential system is $P_1{}' = P_1 \cdot A$, where P_1 is the left transition probability vector $P_1 = \begin{pmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) \end{pmatrix}$. Solving this system for the differential equation that $P_{11}(t)$ satisfies we obtain $P_{11}(t)''' + 6P_{11}{}'' + 11P_{11}{}' = 0$, with the initial conditions $P_{11}(0) = 1, P_{12}(0) = P_{13}(0) = 0$ implying additionally that $P_{11}(0)' = -1, P_{11}(0)'' = 1$.

Taking the Laplace transform of this equation we obtain

$$\begin{aligned} & s^3\mathcal{L}(P_{11}) - s^2 P_{11}(0) - sP_{11}(0)' - P_{11}(0)'' \\ + \; & 6 \cdot (s^2\mathcal{L}(P_{11}) - sP_{11}(0) - P_{11}(0)') \\ + \; & 11 \cdot (s\mathcal{L}(P_{11}) - P_{11}(0)) = 0. \end{aligned}$$

Using the initial values and solving for $\mathcal{L}(P_{11})$ yields $\mathcal{L}(P_{11}) = \frac{s^2+5s+6}{s^3+6s^2+11s}$.

Doing a partial fraction decomposition and inverting,

$$P_{11} = \frac{6}{11} + \frac{5}{11}e^{-3t}\cos(\sqrt{2}t) + \frac{10\sqrt{2}}{11}e^{-3t}\sin(\sqrt{2}t).$$

The invariant measure here is a normalized left eigenvector for the eigenvalue 0 of the differential system. Solving, the unique invariant measure is

$$\rho_0 = \begin{pmatrix} 6/11 & 3/11 & 2/11 \end{pmatrix}.$$

□

3. **Problem:** Consider the continuous time Markov chain on the nonnegative integers which moves from state n to state $n+1$ at rate r_n. Suppose it starts in state 0. What is the characteristic function of the time, τ_n, that the chain first reaches state n? Invert the characteristic function to find the distribution of the arrival time for $n = 3$. Assume r_j distinct.

   Solution: The Markov chain moving from state n to state $n+1$ at rate r_n implies that the waiting time σ_n for the transition from state n to state $n+1$ has an exponential distribution with parameter r_n. Now $\tau_n = \sigma_0 + \ldots + \sigma_{n-1}$ and the σ_n are independent, therefore it's expeditious here to take a Fourier (1,1) transform to exploit the fact that a sum of independent distributions becomes a product of characteristic functions under this transform.

   The Fourier (1,1) transform of an exponential distribution with a probability density function $p(\lambda) = re^{-r\lambda}$ is $\mathcal{F}(t) = r/(r - it)$. We now have three cases.

(a) The r_n are distinct. This is the easiest case. Here $\mathcal{F}(\tau_n) = \prod_{i=0}^{n-1} r_i/(r_i - it)$, and its partial fraction decomposition is found to be

$$\mathcal{F}(\tau_n) = \sum_{i=0}^{n-1} \left(r_i \prod_{\substack{j=0\\ i\neq j}}^{n-1} r_j/(r_j - r_i) \right) 1/(r_i - it).$$

We can use the fact that the transform process is unique to invert this to

$$\tau_n(\lambda) = \sum_{i=0}^{n-1} \left(r_i \prod_{\substack{j=0\\ i\neq j}}^{n-1} r_j/(r_j - r_i) \right) e^{-r_i\lambda}.$$

For the particular case of $n = 3$, we get

$$\tau_3(\lambda) = \frac{r_0 r_1 r_2}{(r_1 - r_0)(r_2 - r_0)} e^{-r_0\lambda} + \frac{r_0 r_1 r_2}{(r_0 - r_1)(r_2 - r_1)} e^{-r_1\lambda} + \frac{r_0 r_1 r_2}{(r_0 - r_2)(r_1 - r_2)} e^{-r_2\lambda}.$$

(b) Exactly two of r_0, r_1, r_2 are equal. Without loss of generality let $r_0 \neq r_1 = r_2 = r$, then $\mathcal{F}(\tau_3) = r_0 r^2/((r_0 - it)(r - it)^2)$. This has the partial fraction decomposition

$$\mathcal{F}(\tau_3) = \frac{r_0 r^2}{(r - r_0)^2 (r_0 - it)} - \frac{r_0 r^2}{(r - r_0)^2 (r - it)} + \frac{r_0 r^2}{(r_0 - r)(r - it)^2}.$$

Inverting the first two terms can be done by invoking uniqueness; to invert the remaining term we can evaluate a contour intergral over the region C_R which is the (clockwise) lower semicircle of radius R. By Cauchy's residue formula we have that for $R > r$

$$\oint_{C_R} \frac{e^{-it\lambda}}{(r - it)^2} = \oint_{C_R} \frac{(1/i)^2 e^{-it\lambda}}{(t - r/i)^2} = -(2\pi i)(-i\lambda)(1/i)^2 e^{-r\lambda} = 2\pi\lambda e^{-r\lambda}.$$

Note here that this contour has a winding number of -1. Letting $R \to \infty$ and observing that the contour integral vanishes on the lower part of the contour, we get that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-it\lambda}}{(r - it)^2} = \frac{2\pi}{2\pi} \lambda e^{-r\lambda} = \lambda e^{-r\lambda},$$

which is the (1,1) Fourier inverse of $(r - it)^{-2}$. Applying this, we finally get that

$$\tau_3 = \frac{r_0 r^2}{(r - r_0)^2} e^{-r_0\lambda} - \frac{r_0 r^2}{(r - r_0)^2} e^{-r\lambda} + \frac{r_0 r^2}{(r_0 - r)} \lambda e^{-r\lambda}.$$

(c) All three of r_0, r_1, r_2 are equal. Let $r = r_0 = r_1 = r_2$, then $\mathcal{F}(\tau_3) = r^3/(r - it)^3$. Using the same contour as above we get that

$$\oint_{C_R} \frac{e^{-it\lambda}}{(r - it)^3} = \oint_{C_R} -\frac{(1/i)^3 e^{-it\lambda}}{(t - r/i)^3} = (2\pi i)/2!(-i\lambda)^2 (1/i)^3 e^{-r\lambda} = \pi\lambda^2 e^{-r\lambda},$$

and so the (1,1) Fourier inverse of $(r-it)^{-3}$ is

$$\frac{1}{2!}\lambda^2 e^{-r\lambda}.$$

Applying this, the final result is

$$\tau_3 = \frac{1}{2!}r^3\lambda^2 e^{-r\lambda}.$$

More generally, if $r_i = r$ for all $i < n$, then

$$\tau_n = \frac{1}{(n-1)!}r^n\lambda^{n-1}e^{-r\lambda}.$$

Note that these are gamma distributions.

□

4. **Problem:** Let $S = \{1,2,3\}$. Consider the function $f : S \to S$, with $f(1) = 1$, $f(2) = f(3) = 2$. Invent a transition matrix p_{ij} for a Markov chain $X_0, X_1, \ldots$ on S, for which $Y_n = f(X_n)$ is *not* a Markov chain and another transition matrix p_{ij} for which Y *is* a Markov chain. What is the condition for which Y will be a Markov chain?

   Solution: For an example of a Markov chain $X = \{X_i\}$ for which $Y = \{Y_i\}$ is not a Markov chain, let X have a transition matrix with $p_{12} = 1, p_{23} = 1, p_{31} = 1$, with all other $p_{ij} = 0$. Now observe that $P(Y_2 = 1 \mid Y_0 = 1, Y_1 = 2) = 0$, as here we must have $X_1 = 2$. However, we also have that $P(Y_2 = 1 \mid Y_0 = 2, Y_1 = 2) = 1$, as here we must have $X_0 = 2$ and $X_1 = 3$. Thus Y can't be Markovian.

   For an example of a Markov chain X for which Y is a Markov chain, let X have a transition matrix with $p_{ii} = 1$ for $i = 1,2,3$, and with all other $p_{ij} = 0$. Note that this is the trivial Markov chain with every state an absorbing state. Here easily $P(Y_{n+1} = j \mid Y_0 = i, \ldots, Y_n = i) = P(Y_{n+1} = j \mid Y_n = i) = [i = j]$, and so Y is Markovian. Indeed, here $f(X)$ is Markovian for any mapping f.

   I claim that the condition for which Y will be a Markov chain is precisely $p_{21} = p_{31}$. Let the initial distribution of X be given by π_0.

   That it's necessary follows from considering

   $$P(Y_1 = 1 \mid Y_0 = 2) = (\pi_0(2)p_{21} + \pi_0(3)p_{31})/(\pi_0(2) + \pi_0(3)).$$

   This must be independent of the initial distribution, and so from the distributions $\pi_0(2) = 1$ and $\pi_0(3) = 1$ we conclude that $p_{21} = p_{31}$.

   To prove sufficiency, consider

   $$p = P(Y_{n+1} = y_{n+1} \mid Y_0 = y_0, \ldots, Y_n = y_n).$$

If $Y_{n+1} = 1$ and $Y_n = 1$, $p = p_{11}$; if $Y_{n+1} = 2$ and $Y_n = 1$, $p = p_{12} + p_{13} = 1 - p_{11}$. If $Y_{n+1} = 1$ and $Y_n = 2$,

$$p = \frac{p_{21} \cdot P(X_n = 2 \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) + p_{31} \cdot P(X_n = 3 \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1})}{P(X_n = 2 \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) + P(X_n = 3 \mid Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1})}.$$

Since $p_{21} = p_{31}$, this simplifies to $p = p_{21} = p_{31}$. It now follows that that if $Y_{n+1} = 2$ and $Y_n = 2$ we must have that $p = 1 - p_{21} = 1 - p_{31}$, and so Y is Markovian.

More generally, the necessary and sufficient condition for Y to be Markovian is that for every $j \in f(S)$,

$$\sum_{f^{-1}(j)} p_{i\cdot} = \sum_{f^{-1}(j)} p_{i'\cdot}$$

for all i, i' such that $f(i) = f(i')$. Mutatis mutandis, the argument above applies to this more general case as well.

$\square$