Applied Stochastic Processes

Chris Long

February 17, 2003

1. **Problem:** Show that the set of all pairs of positive integers can be placed into one-one correspondence with the positive integers by giving an explicit one-one mapping between the two sets.

   Solution: This can be done expediently using the theory of partial difference equations. A standard diagonalization can be characterized by the following relations:

   (a) $f(1,1) = 1$
   (b) $f(x,y) = f(x-1,y) + x + y$
   (c) $f(x,y) = f(x,y-1) + x + y - 1$.

   These can be written in difference notation as:

   (a) $f(1,1) = 1$
   (b) $\frac{\Delta f}{\Delta x} = x + y$
   (c) $\frac{\Delta f}{\Delta y} = x + y - 1$.

   Equation (b) gives $\frac{\Delta^2 f}{\Delta x \Delta y} = 1$ and equation (c) gives $\frac{\Delta^2 f}{\Delta y \Delta x} = 1$, so this is an exact partial difference equation. Summing using equation (b),

   $$f(x,y) = x(x+1)/2 + xy + g(y).$$

   Differencing and setting this equal to the 3rd equation,

   $$x + \frac{\Delta g}{\Delta y} = x + y - 1,$$

   and so $g(y) = y(y+1)/2 - y + C$. Finally $f(1,1) = 1$ implies that $C = -1$, and so

   $$f(x,y) = x(x+1)/2 + xy + y(y+1)/2 - y - 1.$$

   □

2. **Problem:** Let X and Y denote random variables defined on the same probability space which are independent and Poisson distributed with means a and b, respectively.

   (a) Show that the sum $Z = X + Y$ has a Poisson distribution. What is the mean and variance of Z?

   (b) Can the product $W = X \times Y$ ever have a Poisson distribution? What is the mean and variance of W?

Solution:

(a) We have that

$$P(Z = n) = \sum_{i=0}^{n} P(X = i)P(Y = n - i) = \sum_{i=0}^{n} e^{-a}\frac{a^i}{i!} \cdot e^{-b}\frac{b^{n-i}}{(n-i)!}$$

$$= e^{-a-b}\sum_{i=0}^{n} \frac{a^i}{i!}\frac{b^{n-i}}{(n-i)!} = e^{-a-b}\frac{(a+b)^n}{n!}.$$

It follows that Z has a Poisson distribution with mean $a + b$. Since Z is Poisson, the variance is also $a + b$.

(b) We have that

$$E(X \times Y) = E(X) \cdot E(Y) = ab$$

as X and Y are independent. Furthermore,

$$V(X \times Y) = E((X \times Y)^2) - E(X \times Y)^2$$

$$= E(X^2)E(Y^2) - E(X)^2E(Y)^2$$

$$= (E(X^2) - E(X)^2)(E(Y^2) - E(Y)^2) + E(X^2)E(Y)^2 + E(X)^2E(Y^2) - 2E(X)^2E(Y)^2$$

$$= V(X)V(Y) + E(Y)^2(E(X^2) - E(X)^2) + E(X)^2(E(Y^2) - E(Y)^2)$$

$$= V(X)V(Y) + E(Y)^2V(X) + E(X)^2V(Y)$$

$$= ab + b^2a + a^2b = ab(a + b + 1).$$

Now Z cannot be Poisson unless $ab = ab(a + b + 1)$, which would imply that at least one of $a = 0$ or $b = 0$, which is the trivial case.

□

3. **Problem:** Let S be a Poisson set $[0, \infty)$ of rate 1. Let J^{θ_n}, $n = 1, \ldots$, denote the intervals $J^{\theta_n} = (n^2, n^2 + \theta \log(n))$ for each $\theta > 0$. Find

$$F(\theta) = P(\nu(J^{\theta_n}) = 0) \text{ i.o.}),$$

where $\nu(B) = \#(S \cap B)$, the number of elements of S inside B.

Solution: More generally, let S be a Poisson set $[0, \infty)$ of rate a. Then

$$P(\nu(J^{\theta_n}) = 0) = e^{-a\theta \log(n)} = n^{-a\theta}.$$

By the integral test

$$\sum_{n=1}^{\infty} P(\nu(J^{\theta_n}) = 0) = \sum_{n=1}^{\infty} n^{-a\theta}$$

converges if $a\theta > 1$ and diverges to infinity otherwise. The Borel-Cantelli lemmas, together with the stochastic independence of the J^{θ_n}, now imply that $F_a(\theta) = 0$ if $\theta > \frac{1}{a}$ and $F_a(\theta) = 1$ if $\theta \leq \frac{1}{a}$. □

4. **Problem:** Let S be a Poisson random set with rate a in the plane. Show there is a triangle with one vertex at the origin having arbitrarily large area and yet having *no* points of S inside it.

   Solution: Let S_n be non-overlapping open circular sectors centered at the orgin, with S_n having angle $2\pi/2^n$ and radius $\sqrt{(\log(n)2^n)\theta/\pi}$. Thus $|S_n| = \theta \log(n)$, and so if we choose $0 < \theta \leq \frac{1}{a}$ the same analysis as in the previous problem shows that with probability 1 infinitely many of the S_n contain no points of S. If follows that with probability 1 there exist triangles with one vertex at the origin having arbitrarily large area, yet containing no points of S. □