

49th Putnam Competition

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Problems

Morning

- A-1. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.
- A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a non-zero function g defined on (a, b) such that the wrong product rule is true for x in (a, b) .
- A-3. Determine, with proof, the set of real numbers x for which $\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc\left(\frac{1}{n}\right) - 1\right)^x$ converges.
- A-4. Justify your answers.
- (a) If every point on the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
 - (b) What if “three” is replaced by “nine”?
- A-5. Prove that there exists a *unique* function f from the set \mathbb{R}^+ of positive real numbers to \mathbb{R}^+ such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all $x > 0$.
- A-6. If a linear transformation A on an n -dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

Afternoon

- B-1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y , and z positive integers.
- B-2. Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.
- B-3. For every n in the set $\mathbb{Z}^+ = \{1, 2, \dots\}$ of positive integers, let $r(n)$ be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r(n) \leq g$ for all n in \mathbb{Z}^+ .
- B-4. Prove that if $\sum_{n=1}^{\infty} a(n)$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a(n))^{n/(n+1)}$.
- B-5. For positive integers n , let $M(n)$ be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1 . Find, with proof, the rank of $M(n)$. (According to the definition the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with non-zero determinant.)

One may note that

$$M(1) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \text{ and } M(2) = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

- B-6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at + b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t(n) = n(n+1)/2$ with n in $\{0, 1, 2, \dots\}$).

Solutions

- A-1. The area is 6; the graph I leave to the reader.
- A-2. We find all such functions. Note that $(fg)' = f'g' \Rightarrow f'g' = f'g + fg'$ hence if $g(x), f'(x) - f(x) \neq 0$ we get that $g'(x)/g(x) = f'(x)/(f'(x) - f(x))$. For the particular f given, we then get that $g'(x)/g(x) =$

$(2x)e^{x^2}/((2x-1)(e^{x^2})) \Rightarrow g'(x)/g(x) = 2x/(2x-1)$ (since $e^{x^2} > 0$). Integrating, we deduce that $\ln |g(x)| = x + (1/2) \ln |2x-1| + c$ (an arbitrary constant) $\Rightarrow |g(x)| = e^c \sqrt{|2x-1|} e^x \Rightarrow g(x) = C \sqrt{|2x-1|} e^x$, C arbitrary $\neq 0$. We finish by noting that any $g(x)$ so defined is differentiable on any open interval that does not contain $1/2$.

A-3. The answer is $x > \frac{1}{2}$. To see this, note that by Taylor's theorem with remainder $\sin(\frac{1}{n}) = \sum_{i=1}^{k-1} (-1)^{i-1} \frac{1}{i!} n^{-(2i+1)} + c(-1)^{k-1} \frac{1}{k!} n^{-(2k+1)}$, where $0 \leq c \leq \frac{1}{n}$. Hence for $n \geq 1$ $(1/n)/(1/n - 1/(3!n^3) + 1/(5!n^5)) - 1 < (1/n) \csc(1/n) - 1 < (1/n)/(1/n - 1/(3!n^3)) - 1 \Rightarrow$ for n large enough, $(1/2)1/(3!n^2) < (1/n) \csc(1/n) - 1 < 2 \cdot 1/(3!n^2)$. Applying the p-test and the comparison test, we see that $\sum_{n=1}^{\infty} (\frac{1}{n} \csc(\frac{1}{n}) - 1)^x$ converges iff $x > \frac{1}{2}$.

A-4.

- (a) The answer is yes. Assume not and consider two equilateral triangles with side one that have exactly one common face \Rightarrow all points a distance of $\sqrt{3}$ apart are the same color; now considering a triangle with sides $\sqrt{3}, \sqrt{3}, 1$ we reach the desired contradiction.
- (b) In this case, there does not necessarily exist two points of the same color exactly one inch apart; this can be demonstrated by considering a tessellation of the plane by a 3×3 checkerboard with side 2, with each component square a different color (color of boundary points chosen in an obvious manner).

The length of the side of the checkerboard is not critical (the reader may enjoy showing that $3/2 < \text{side} < 3\sqrt{2}/2$ works).

A-5. Solution 1:

Clearly $f(x) = 2x$ is one such solution; we need to show that it is the *only* solution. Let $f^1(x) = f(x)$, $f^n(x) = f(f^{n-1}(x))$ and notice that $f^n(x)$ is defined for all $x > 0$. An easy induction establishes that for $n > 0$, $f^n(x) = a_n x + b_n f(x)$, where $a_0 = 0$, $b_0 = 1$ and $a_{n+1} = 6b_n$, $b_{n+1} = a_n - b_n \Rightarrow b_{n+1} = 6b_{n-1} - b_n$. Solving this latter equation in the standard manner, we deduce that $\lim_{n \rightarrow \infty} a_n/b_n = -2$, and since we have that $f^n(x) > 0$ and since b_n is alternately negative and positive; we conclude that $2x \leq f(x) \leq 2x$ by letting $n \rightarrow \infty$.

Solution 2: (Dan Bernstein, Princeton)

As before, $f(x) = 2x$ works. We must show that if $f(x) = 2x + g(x)$ and f satisfies the conditions then $g(x) = 0$ on \mathbb{R}^+ . Now $f(f(x)) =$

$6x - f(x)$ means that $2f(x) + g(f(x)) = 6x - 2x - g(x)$, i.e., $4x + 2g(x) + g(f(x)) = 4x - g(x)$, i.e., $3g(x) + g(f(x)) = 0$. This then implies $g(f(f(x))) = 9g(x)$. Also note that $f(x) > 0$ implies $g(x) > -2x$. Suppose $g(x)$ is not 0 everywhere. Pick y at which $g(y) \neq 0$. If $g(y) > 0$, observe $g(f(y)) = -3g(y) < 0$, so in any case there is a y_0 with $g(y_0) < 0$. Now define $y_1 = f(f(y_0)), y_2 = f(f(y_1))$, etc. We know $g(y_{n+1})$ equals $g(f(f(y_n))) = 9g(y_n)$. But $y_{n+1} = f(f(y_n)) = 6y_n - f(y_n) < 6y_n$ since $f > 0$. Hence for each n there exists $y_n < 6^n y_0$ such that $g(y_n) = 9^n g(y_0)$. The rest is obvious: $0 > g(y_0) = 9^{-n} g(y_n) > -2 \cdot 9^{-n} y_n > -2(6/9)^n y_0$, and we observe that as n goes to infinity we have a contradiction.

A-6. The answer is yes. First note that if x_1, \dots, x_{n+1} are the eigenvectors, then we must have that $a_{n+1}x_{n+1} = a_1x_1 + \dots + a_nx_n$ for some non-zero scalars a_1, \dots, a_{n+1} . Multiplying by A on the left we see that $\lambda_{n+1}a_{n+1}x_{n+1} = \lambda_1a_1x_1 + \dots + \lambda_na_nx_n$, where λ_i is the eigenvalue corresponding to the eigenvectors x_i . But since we also have that $\lambda_{n+1}a_{n+1}x_{n+1} = \lambda_{n+1}a_1x_1 + \dots + \lambda_{n+1}a_nx_n$ we conclude that $\lambda_1a_1x_1 + \dots + \lambda_na_nx_n = \lambda_{n+1}a_1x_1 + \dots + \lambda_{n+1}a_nx_n \Rightarrow a_1(\lambda_1 - \lambda_{n+1})x_1 + \dots + a_n(\lambda_n - \lambda_{n+1})x_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_{n+1} = \lambda$ since x_1, \dots, x_n are linearly independent. To finish, note that the dimension of the eigenspace of λ is equal to n , and since this equals the dimension of the nullspace of $A - \lambda I$ we conclude that the rank of $A - \lambda I$ equals $n - n = 0 \Rightarrow A - \lambda I = 0$.

B-1. Let $x = a - 1, y = b - 1, z = 1$; we then get that $xy + xz + yz + 1 = (a - 1)(b - 1) + a - 1 + b - 1 + 1 = ab$.

B-2. The statement is true. If $x + 1 \geq 0$ we have that $\sqrt{y(y+1)} - 1 \leq x \Rightarrow x^2 \geq y^2 + y + 1 - 2\sqrt{y^2 + y} \geq y^2 - y$ since $2y + 1 \geq 2\sqrt{y^2 + y}$ since $(2y + 1)^2 \geq 4(y^2 + y)$ if $y \geq 0$. If $x + 1 < 0$, we see that $\sqrt{y(y+1)} \leq -x - 1 \Rightarrow x^2 \geq y^2 + y + 1 + 2\sqrt{y^2 + y} \geq y^2 - y$.

B-3. The answer is $(1 + \sqrt{3})/2$. We write $|c - d\sqrt{3}|$ as $|(n - d) - d\sqrt{3}|$; I claim that the minimum over all $d, 0 \leq d \leq n$, occurs when $d = e = \lfloor n/(1 + \sqrt{3}) \rfloor$ or when $d = f = e + 1 = \lfloor n/(1 + \sqrt{3}) \rfloor + 1$. To see this, note that $(n - e) - e\sqrt{3} > 0$ and if $e' < e$, then $(n - e') - e'\sqrt{3} > (n - e) - e\sqrt{3}$, and similarly for $f' > f$. Now let $r = n/(1 + \sqrt{3}) - \lfloor n/(1 + \sqrt{3}) \rfloor$ and note that $|(n - e) - e\sqrt{3}| = r(1 + \sqrt{3})$ and $|(n - f) - f\sqrt{3}| = (1 - r)(1 + \sqrt{3})$. Clearly one of these will be $\leq (1 + \sqrt{3})/2$. To see that $(1 + \sqrt{3})/2$ cannot be lowered, note that since $1 + \sqrt{3}$ is irrational, r is uniformly distributed mod 1.

Remark: We do not really need the result that x irrational $\Rightarrow xn - \lfloor xn \rfloor$ u. d. mod 1, it would suffice to show that x irrational $\Rightarrow xn - \lfloor xn \rfloor$ is dense in $(0, 1)$. But this is obvious, since if x is irrational there exists arbitrarily large q such that there exists p with $(p, q) = 1$ such that $p/q < x < (p + 1)/q$. The nifty thing about the u. d. result is that it answers the question: what number x should we choose such that the density of $\{n : r(n) < x\}$ equals $t, 0 < t < 1$? The u. d. result implies that the answer is $t(1 + \sqrt{3})/2$. The u. d. result also provides the key to the question: what is the average value of $r(n)$? The answer is $(1 + \sqrt{3})/4$.

B-4. Note that the subseries of terms $a(n)^{\frac{n}{n+1}}$ with $a(n)^{\frac{1}{n+1}} \leq \frac{1}{2}$ converges since then $a(n)^{\frac{n}{n+1}}$ is dominated by $1/2^n$, the subseries of terms $a(n)^{\frac{n}{n+1}}$ with $a(n)^{\frac{1}{n+1}} > \frac{1}{2}$ converges since then $a(n)^{\frac{n}{n+1}}$ is dominated by $2a(n)$, hence $\sum_{n=1}^{\infty} a(n)^{\frac{n}{n+1}}$ converges.

B-5. Solution 1: Since $M(n)$ is skew-symmetric, $M(n)$ is singular for all n , hence the rank can be at most $2n$. To see that this is indeed the answer, consider the submatrix $M_i(n)$ obtained by deleting row i and column i from $M(n)$. From the definition of the determinant we have that $\det(M_i(n)) = \sum (-1)^{\delta(k)} a_{1k(1)} \cdots a_{(2n)k(2n)}$, where k is member of S_{2n} (the group of permutations on $\{1, \dots, 2n\}$) and $\delta(k)$ is 0 if k is an even permutation or 1 if k is an odd permutation. Now note that $(-1)^{\delta(k)} a_{1k(1)} \cdots a_{(2n)k(2n)}$ equals either 0 or ± 1 , and is non-zero iff $k(i) \neq i$ for all i , i.e. iff k has no fixed points. If we can now show that the set of all elements k of S_{2n} , with $k(i) \neq i$ for all i , has odd order, we win since this would imply that $\det(M_i(n))$ is odd $\Rightarrow \det(M_i) \neq 0$. To show this, let $f(n)$ equal the set of all elements k of S_n with $k(i) \neq i$ for all i . We have that $f(1) = 0, f(2) = 1$ and we see that $f(n) = (n-1)(f(n-1) + f(n-2))$ by considering the possible values of $f(1)$ and whether or not $f(f(1)) = 1$; an easy induction now establishes that $f(2n)$ is odd for all n .

Remark: In fact, it is a well-known result that $f(n) = n!(1/2! - 1/3! + \cdots + (-1)^n/n!)$.

Solution 2: As before, since $M(n)$ is skew-symmetric $M(n)$ is singular for all n and hence can have rank at most $2n$. To see that this is the rank, let $M_i(n)$ be the submatrix obtained by deleting row i and column i from $M(n)$. We finish by noting that $M_i(n)^2 \equiv I_{2n} \pmod{2}$, hence $M_i(n)$ is nonsingular.

B-6. Call a pair of integers (a, b) a *triangular pair* if $at + b$ is a triangular number iff t is a triangular number. I claim that $(9, 1)$ is a triangular pair. Note that $9(n(n+1)/2) + 1 = (3n+1)(3n+2)/2$ hence $9t + 1$ is triangular if t is. For the other direction, note that if $9t + 1 = n(n+1)/2 \Rightarrow n = 3k+1$ hence $9t+1 = n(n+1)/2 = 9(k(k+1)/2)+1 \Rightarrow t = k(k+1)/2$, therefore t is triangular. Now note that if (a, b) is a triangular pair then so is $(a^2, (a+1)b)$, hence we can generate an infinite number of triangular pairs starting with $(9, 1)$.

Remark: The following is a proof of necessary and sufficient conditions for (a, b) to be a triangular pair.

I claim that (a, b) is a triangular pair iff for some odd integer o we have that $a = o^2, b = (o^2 - 1)/8$. I will first prove the direction \Leftarrow . Assume we have $a = o^2, b = (o^2 - 1)/8$. If $t = n(n+1)/2$ is any triangular number, then the identity $o^2n(n+1)/2 + (o^2 - 1)/8 = (on + (o-1)/2)(on + (o+1)/2)/2$ shows that $at + b$ is also a triangular number. On the other hand if $o^2t + (o^2 - 1)/8 = n(n+1)/2$, the above identity implies we win if we can show that $(n - (o-1)/2)/o$ is an integer, but this is true since $o^2t + (o^2 - 1)/8 \equiv n(n+1)/2 \pmod{o^2} \Rightarrow 4n^2 + 4n \equiv -1 \pmod{o^2} \Rightarrow (2n+1)^2 \equiv 0 \pmod{o^2} \Rightarrow 2n+1 \equiv 0 \pmod{o} \Rightarrow n \equiv (o-1)/2 \pmod{o}$. For the direction \Rightarrow assume that (a, b) and $(a, c), c \geq b$, are both triangular pairs; to see that $b = c$ notice that if $at + b$ is triangular for all triangular numbers t , then we can choose t so large that if $c > b$ then $at + c$ falls between two consecutive triangular numbers; contradiction hence $b = c$. Now assume that (a, c) and (b, c) are both triangular pairs; I claim that $a = b$. But this is clear since if (a, c) and (b, c) are triangular pairs $\Rightarrow (ab, bc + c)$ and $(ab, ac + c)$ are triangular pairs $\Rightarrow bc + c = ac + c$ by the above reasoning $\Rightarrow bc = ac \Rightarrow$ either $a = b$ or $c = 0 \Rightarrow a = b$ since $c = 0 \Rightarrow a = b = 1$. For a proof of this last assertion, assume $(a, 0), a > 1$, is a triangular pair; to see that this gives a contradiction note that if $(a, 0)$ is a triangular pair $\Rightarrow (a^2, 0)$ is also triangular pair, but this is impossible since then we must have that $a(a^3 + 1)/2$ is triangular (since $a^2a(a^3 + 1)/2$ is triangular) but $(a^2 - 1)a^2/2 < a(a^3 + 1)/2 < a^2(a^2 + 1)/2$ (if $a > 1$). We are now done, since if (a, b) is a triangular pair $\Rightarrow a0 + b = n(n+1)/2$ for some $n \geq 0 \Rightarrow b = ((2n+1)^2 - 1)/8$.