Self-dual Vertex Operator Superalgebras and the Baby Monster

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Chapter 1

Introduction

At the end of the 1970s there was found a surprising and at first seemingly inexplicable connection between Fourier coefficients of modular functions and dimensions of irreducible representations of the largest sporadic finite simple group, the Monster. A partial explanation of these so-called "moonshine" phenomena was found in the 1980s with the help of vertex operator algebras. In the present work self-dual vertex operator superalgebras will be studied; these are vertex operator superalgebras with exactly one irreducible representation. The vertex operator superalgebras have by the theory of modular functions and finite groups connections to many further structures in mathematics and physics, like affine Kac-Moody algebras, elliptic genera, operads and conformal quantum field theories. A central theme of this work is the analogy between *codes, lattices and vertex operator algebras*, which also motivates the following theorem, one of the main results of this work:

Theorem 1 *There exists a vertex operator superalgebra $VB^\natural$ of rank $23\frac{1}{2}$ for which the Baby Monster B - the second largest sporadic finite simple group - operates in a natural way. It possesses the character*

$$\chi_{VB^\natural} = \chi_{1/2}^{47} - 47\chi_{1/2}^{23} = q^{-\frac{47}{48}}(1 + 4371q^{3/2} + 96256q^2 + 1143745q^{5/2} + \ldots),$$

where $\chi_{1/2} = \sqrt{\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}/(q^{1/24}\prod_{n=1}^{\infty}(1-q^n))}$.

This Baby Monster vertex operator superalgebra is analogous to the shorter Golay code or the shorter Leech lattice. However, the main concern of this work is more general and consists in the development of a new method towards systematic classification of self-dual vertex operator superalgebras.

Chapter 2

Chapter 3

Chapter 4

The Baby Monster Vertex Operator Superalgebra

The aim of this chapter is the construction of a SVOA $VB^\natural$ of rank $23\frac{1}{2}$ on which the Baby Monster operates via automorphisms. This SVOA is analogous to the shorter Golay code g_{22} (v. [PS75]) or to the shorter Leech lattice O_{23} (v. [CS82]), which has as an automorphism group a double cover of the Matheiu group M_{22} resp. the Conway group Co_2. As g_{22} contains no tetrads (vectors of weight 4) and as O_{23} contains no roots (vectors of squared length 2), therefore $VB^\natural$ contains no nontrivial Lie subalgebras V_1 (vectors of conformal weight 1). The SVOA $VB^\natural$ is the natural object on which $B \times \mathbb{Z}_2$ operates via automorphisms, and in this sense it provides the most natural definition of the Baby Monster.

In Section 4.2 $VB^\natural$ is constructed from the moonshine module $V^\natural$ using the methods of the previous chapter. Additionally, in Section 4.1 a description of Virasoro sub-vertex operator algebras of rank $\frac{1}{2}$ will be given.

4.1 Decomposition of the Moonshine Module $V^\natural$ under $L_{1/2}^{\otimes 48}(0)$

The moonshine module cannot, since $V_1^\natural = 0$, be decomposed as a sum of highest weight representations of affine Kac-Moody algebras. As Dong, Mason and Zhu have shown in [DMZ94], however, one can find 48 pairs of associated commuting sub-VOAs $L_{1/2}(0)$ of rank $\frac{1}{2}$ with $c = \frac{1}{2}$ unitary highest weight representations. The monster therefore decomposes as the direct sum of VOA-modules of $L_{1/2}^{\otimes 48}(0)$. We will also show in this section that the results from a paper [MN93] of W. Meyer and W. Neutsch on associative subalgebras of the Griess algebra follows. Furthermore, we give with the help of invariant theory a new, more exact description of the decomposition of $V^\natural$.

The 196884-dimensional commutative nonassociative Griess algebra $\mathcal{B}$ [Gri82] is the weight 2 part $V_2^\natural$ of the moonshine module. For two elements $a, b \in \mathcal{B}$.

The structure of associative subalgebras of $\mathcal{B}$ was examined in [MN93].

Theorem 2 *MeyerNeutsch Let $\mathcal{U}$ be a k-dimensional associative subalgebra (the real form) of the Griess algebra $\mathcal{B}$.*

Bibliography

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