SOME RESULTS ON DIGIT-SUM SEQUENCES

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Abstract. Define $f(n)$ to equal n plus the sum of its digits. Call a number m a "starter" if there does not exist a k such that $m = f(k)$. We demonstrate that there are infinitely many starters of the form $10^n + b,\ b \geq 1$, if and only if $b - 1$ is a starter.

Define $f(n)$ to equal n plus the sum of its digits. Call a number m a "starter" if there does not exist a k such that $m = f(k)$. For example, the first such sequence is $1, 2, 4, 8, 16, 23, 28, \ldots$. The third sequence is $5, 10, 11, 13, 17, \ldots$. Stated as a problem by Rubin [1], it was subsequently shown that no integer less than 100 occurs in more than one sequence [2]. It was also shown that there are infinitely many such sequences.

Recently, however, in the problem section of this journal [3], the question of existence of infinitely many starters of the form $10^n + 122$ came up. It was shown that for $n = 15$, $10^n + 122$ was *not* a starter: 1000000000000122 already appears, as the next term, in the sequence containing 999999999999993. In this article, I will demonstrate that there are infinitely many starters of the form $10^n + b$, $b \geq 1$, if and only if $b - 1$ is a starter.

Lemma 1. *If $b - 1$, $b > 0$, is not a starter then only finitely many n exist such that $10^n + b$ is a starter.*

Proof. If $n > \log(b)$ we have that $f(10^n + c) = 10^n + b$, where $f(c) = b - 1$, hence only finitely many such n can exist. □

We now prove a rather surprising result which states that, under certain conditions, we can generate a (much) larger starter from a given starter.

Lemma 2. *If $b - 1$ and $10^n + b$ are starters, where $n > \max(1, \log(b))$, then $10^N + b$ is also a starter, where $N = 10^n + n + 1$.*

Proof. Note that since $b-1$ is a starter we can find no positive c such that $f(10^N + c) = 10^N + 1 + f(c) = 10^N + b$. If x started with a digit less than 9, we would have that $f(x) \leq f(9 \cdot 10^{N-1} - 1) = 9 \cdot 10^{N-1} - 1 + 8 + 9(N-1) < 10^N + b$. We now see that if $f(x) = 10^N + b$, x must start with $m > 0$ 9's.

Let $x = 10^N - 10^m + c$, $0 \leq c \leq 9 \cdot 10^{m-1} - 1$. We now consider three separate cases.

If $m \leq n$ we see that $10^N + b < 10^N - 10^n + 9(N-n) \leq 10^N - 10^m + 9(N-m) = f(10^N - 10^m) \leq f(x)$; contradiction.

If $m \geq n+2$ we see that $10^N + b > 10^N - 10^{n+1} - 1 + 9(N-1) + 8 \geq 10^N - 10^{m-1} - 1 + 9(N-1) + 8 = f(10^N - 10^{m-1} - 1) \geq f(x)$; contradiction.

If $m = n+1$ we see that $10^N + b = f(x) = 10^N - 10^{n+1} + f(c) + 9(N-(n+1)) = 10^N - 10^{n+1} + f(c) + 9 \cdot 10^n$ implies $10^n + b = f(c)$; contradiction. □

I find this result interesting because of the rapid growth of the numbers involved. For example, note that $10^3 + 122$ is a starter; from Lemma 2 we know that $10^{10^3+3+1} + 122 = 10^{1004} + 122$ is also a starter. This is already larger than a googol! The next starter that Lemma 2 would generate is larger than a googolplex, and the next is larger than Skewe's number.

We need one more result to prove our main theorem, i.e. we need to demonstrate the existence of a sufficiently large initial starter.

Lemma 3. *Let $m = \lfloor \log(b) \rfloor + 1$; if $b-1$ is a starter and $b \geq 18$, then $10^m + b$ is a starter.*

Proof. Assume that for some c we had that $f(10^m + c) = 10^m + b$. This implies that $10^m + f(c) + 1 = 10^m + b$, but this is impossible since $b-1$ is a starter. If c were negative, we would have that $f(10^m + c) \leq f(10^m - 1) = 10^m - 1 + 9m$, and this latter term is strictly less than $10^m + b$ if $b \geq 18$. □

We now have enough tools to prove our main theorem.

Theorem 1. *There are infinitely many starters of the form $10^n + b$, $b > 0$, if and only if $b-1$ is a starter.*

Proof. By Lemma 1 we know that if there are infinitely many starters of the form $10^n + b$ then $b-1$ must be a starter. Now assume that $b-1$ is a starter. If $b \geq 18$, we know by Lemma 3 that $10^m + b$ is a starter, and since $m > \log(b) > 1$ we can exhibit infinitely many starters of the required form by iterating the procedure in Lemma 2. To finish the proof, note that the only starters less than 17 are $1, 3, 5, 7$, and 9; it is easily verified that $102, 104, 1006, 10008$, and 110 are starters and we can again iterate the procedure in Lemma 2. □

References

[1] F. Rubin, Problem 1078 - Digit Sum Sequences, *Journal of Recreational Mathematics*, **14:2**(1981-82), 141-142.

[2] V. G. Feser, Solution to Problem 1078, *Journal of Recreational Mathematics*, **15:2**(1982-83), 155-156.

[3] H. L. Nelson, Commentary to the Solution to Problem 1078, *Journal of Recreational Mathematics*, **20:4**(1988), 304-305.

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