Differential Equations 244
Sample Final Exam

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1. (20) Solve the equations:

   (a) (10)

   $$3t^2y + 8ty^2 + (t^3 + 8t^2y + 12y^2)\frac{dy}{dt} = 0$$

   and,

   (b) (10)

   $$\frac{dy}{dt} + \frac{2t}{1+t^2} \cdot y = \frac{1}{1+t^2}.$$

2. (20) Given the ODE,

   $$\frac{dy}{dx} = x + y.$$

   (a) (5) Compute the isoclines.

   (b) (9) Sketch the direction field.

   (c) (6) Draw the integral curve passing through the point $(0, 1)$.

3. (20) The temperature of a cup of coffee is measured to be 200^o F. A minute later the temperature is measured to be 190^o F. If the room has a temperature of 70^o F, how long do you have to wait for the temperature to reach 150^o F. Assume that the law governing cooling is given by Newton's law that states that the rate of change of temperature is proportional to the temperature difference with the surrounding temperature. Set up an initial value problem. Solve it and then use the solution to answer the question.

4. (20) Solve:
$$y'' + 4y' + 4y = \sin t + e^{-2t}.$$

5. (20) Solve:
$$y'' + 6y' + 9y = t^{7/2}e^{-3t}.$$

6. (20) Solve the initial value problem,
$$y'' + t^2y' + 2ty = 0, \; y(0) = 1, \; y'(0) = 0.$$

   Make sure that you compute and DISPLAY a suitable recurrence relation for any general power series solution. Furthermore write out your power series solution to the Initial value problem to **four non-zero terms**.

7. (20) Solve:
$$\frac{d^5y}{dx^5} - \frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0.$$

8. (20) Solve the initial value problem for the system:
$$\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}, \;\; \mathbf{x}(0) = \begin{pmatrix} -1 \\ 7 \\ 3 \end{pmatrix}.$$

9. (20) Given the system,
$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}.$$

   (a) (11) Solve for the general solution.

   (b) (7) Draw the phase plane picture/portrait (you only have to sketch a few orbits which gives me an idea that you understand the general nature of the solutions!!).

   (c) (2) Compute the coordinates of the critical points. What sort of stability do the critical points have?

10. (20) Given the ODE:

$$(x^2 + 4x + 5)(e^x - 1)^2(x-1)^2y'' + xy' + y = 0.$$

(a) (6) Locate all the ordinary points and the singular points.

(b) (6) Classify further the singular points as regular singular points and irregular singular points.

(c) (2) Suppose you sought a solution for the displayed ODE as a power series of the form $\sum_{n=0}^{\infty} a_n(x+3)^n$. Are you justified in seeking a solution in this form? In a sentence explain why.

(d) (6) If your answer in part (c) is that a series solution of the type indicated is valid, then find the lower bound for an interval around $x_0 = -3$ where your power series solution will be valid.