Exercises 1-3 pertain to operator convexity: Let $\mathcal{A}$ be a $C^*$ algebra with identity 1, and let $\mathcal{A}^+$ denote the cone of positive elements of of $\mathcal{A}$. A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in case for all strictly positive $a, b \in \mathcal{A}$ and all $t \in [0, 1]$, 
\[ f(ta + (1-t)b) \leq f(a) + (1-t)f(b), \]
using the partial order defined by $\mathcal{A}^+$. (Strictly positive means the spectrum lies in $(0, \infty)$, and not only in $[0, \infty)$).

**Exercise 1:** Show that the function $f(t) = t^{-1}$ is operator convex.

**Exercise 2:** Show that for all $p \in (0, 1)$, function $f(t) = t^p$ is has the integral representation 
\[ t^p = \frac{\sin(\pi p)}{\pi} \int_0^{\infty} s^p \left( \frac{1}{s} - \frac{1}{t+s} \right) ds. \]
Use this to show that $-f$ is operator convex for such $p$.

**Exercise 3:** Use the fact that for $f(t) = t \ln t$, 
\[ f(t) = \lim_{p \to 1} \frac{t^p - t}{p - 1} \]
to show that $f$ is operator convex.

**Exercise 4:** Let $\mathcal{A}$ be a $C^*$ algebra without an identity. Show that for all $a \in \mathcal{A}$, 
\[ \|a\| = \sup_{\|b\| \leq 1} \|ab\|. \]

**Exercise 5:** Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{F}$ be the $*$-algebra of all finite rank operators on $\mathcal{H}$. Compute $\mathcal{F}'$ and $\mathcal{F}''$. Show that if $a \in \mathcal{B}(\mathcal{H})$ with $\|a\| \leq 1$, then there is a sequence $\{a_n\}$ in $\mathcal{F}$ that converges to $a$ in the strong operator topology.

**Exercise 6:** Let $\pi$ and $\sigma$ be two cyclic representations of $C^*$ algebra $\mathcal{A}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, with cyclic vectors $\zeta$ and $\eta$ respectively. Show that if 
\[ \langle \zeta, \pi(a)\zeta \rangle_{\mathcal{H}} = \langle \eta, \sigma(a)\eta \rangle_{\mathcal{K}} \]
then $\pi$ and $\sigma$ are equivalent; i.e., there us a unitary $u : \mathcal{H} \to \mathcal{K}$ such that 
\[ \pi(a)u = u\sigma(a) \]
for all $a \in \mathcal{A}$.