Problem Set 4 for Math 502, March 29, 2013

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1. Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, and let $1 < p < \infty$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mu)$. Suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for almost every $x \in \Omega$. Show that $f \in L^p(\mu)$, and that $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to f in $L^p(\mu)$.

2. Let $(\Omega, \mathcal{S}, \mu)$ be a σ-finite measure space such that $L^2(\mu)$ is separable. Let $\{u\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mu)$.

(a) Show that $\{u_m(x)u_m(y)\}_{m,n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\Omega\times\Omega, \mathcal{S}\otimes S, \mu\times\mu)$. (This comes down to showing that if $K \in L^2(\Omega\times\Omega, \mathcal{S}\otimes S, \mu\times\mu)$ and

$$\int_{\Omega\times\Omega} K(x,y)u_m(x)u_n(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = 0$$

for all m, n, then $K = 0$.)

(b) Let $K \in L^2(\Omega\times\Omega, \mathcal{S}\otimes S, \mu\times\mu)$. Define the numbers $\{a_{m,n}\}_{m,n\in\mathbb{N}}$ by

$$a_{m,n} = \int_{\omega\times\Omega} K(x,y)u_m(x)u_n(y)\mathrm{d}\mu\times\mathrm{d}\mu\ .$$

For $N \in \mathbb{N}$, define the function K_N by

$$K_N(x,y) = \sum_{m,n=1}^{N} a_{m,n}u_m(x)u_n(y)\ .$$

Show that for $\epsilon > 0$, there exists an N_ϵ so that

$$\|K - K_{N_\epsilon}\|_2 < \epsilon$$

where the norm is the $L^2(\mu\times\mu)$ norm.

3. Continue with the notation from the previous problem:

(a) Define the linear operator K on $L^2(\mu)$ by

$$Kf(x) = \int_\Omega K(x,y)f(y)\mathrm{d}\mu\ .$$

Show that K is continuous from $L^2(\mu)$ to $L^2(\mu)$ (in the norm topology). Define K_N in the analogous manner, and show that it, too, is continuous in the norm topology.

(b) Let $\{f_j\}$ be a weakly convergent sequence in $L^2(\mu)$. Show that for each N, $\{K_N f_j\}_{j\in\mathbb{N}}$ is a strongly convergent sequence in $L^2(\mu)$. Use this, and the above results, to show that also $\{Kf_j\}_{j\in\mathbb{N}}$ is a strongly convergent sequence in $L^2(\mu)$.

4. Continue with the notation from the previous problem:

(a) Show that there is a unit vector $f_1 \in L^2(\mu)$ such that

$$\|Kf\|_2 \leq \|f_1\|_2$$

for all f with $\|f\|_2 \leq 1$. That is, there is a unit vector f_1 such that

$$\|Kf_1\|_2 = \|K\|$$

where the norm on the right is the norm of K as an operator from $L^2(\mu)$ to $L^2(\mu)$.

(b) Let $\omega = [0,1]$, and let μ be Lebesgue measure. Consider the operator M on $L^2(\mu)$ define by $Mf(x) = xf(x)$. Show that M is continuous in the norm topology, but there is no unit vector f in $L^2(\mu)$ such that

$$\|Mf\|_2 = \|M\| \ .$$